Early generalization

Part 2

YOUNG CHILDREN SOLVING ADDITIVE STRUCTURE PROBLEMS

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This paper describes a study to analyse how 4-6-year-olds (N=45) children solve different types of additive reasoning problems. Individual interviews were conducted on kindergarten children when solving the problems. Their performance as well as their explanations were analysed when solving additive reasoning problems. The additive reasoning problems comprised simple, inverse and comparative problems. Results suggested that Portuguese kindergarten children have some informal knowledge that allowed them to solve additive structure problems with understanding. Children performed better in the simple additive problems and found the comparative problems more difficult.

INTRODUCTION

In mathematics children are expected to be able to attribute a number to a quantity, which is measuring (Nunes & Bryant, 2010a), but they also are expected to be able to quantify relations. When quantities are measured, they have a numerical value, but it is possible to reason about the quantities without measure them. In agreement with Nunes, Bryant and Watson (2010), it is crucial for children to learn to make both connections and distinctions between number and quantity. Quantitative reasoning results from a quantifying relations and manipulate them (Nunes & Bryant, 2010a), making relationships between quantities valuable (Thompson, 1994). For Nunes and Bryant (2010a), quantifying relations can be done by additive or multiplicative reasoning. Quoting the authors "[…] Additive reasoning tells us about the difference between quantities; multiplicative reasoning tells us about the ratio between quantities." (p.8). In the literature additive reasoning is associated to addition and subtraction (see Vergnaud, 1983) and multiplicative reasoning is associated to multiplication and division problems (see Steffe, 1994; Vergnaud, 1983).

Children can use their informal knowledge to analyse and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes & Bryant, 1996).

ABOUT THE ADDITVE REASONING

Piaget (1952) argued that children's understanding of arithmetical operations arises from their *schema*. A 'schema' is a representation of an action in which only the essential aspects of the action are evident. He identified three schemas related to additive reasoning: joint, separate and one-to-one correspondence. The author pointed out that children are able to master addition and subtraction only when they understand the inverse relation between these operations, which is achieved by the 7-year-olds. More recently, Nunes and Bryant (1996) referred that kindergarten children of 5-6-year-olds can relate their understanding of number as a measure of set size to their conception of addition / subtraction as an increase / decrease in quantities. This can help children to begin to understand that one operation is the inverse of the other. The schema from which children begin to understand addition and subtraction are representations of the act of joint and separate, respectively (Nunes, Campos, Magina & Bryant, 2005). These schemas allow 5-year-olds children to solve a problem such as: ”Anna has 3 candies. Her mother gave her 2 more candies. How many candies does Anna have now?”.

Additive reasoning problems involve one variable and they tell us about the difference between quantities. The part-whole relation is the invariant of the additive reasoning. The whole equals the sum of the parts. Nunes, Bryant and Watson (2010) argue that additive relations are used in one variable problems when quantities of the same kind are put together, separated or compared.

Carpenter and Moser (1982, 1984) presented a classification of addition and subtraction problem that does not characterize all the types of word problems involving additive reasoning, but those who are appropriate for primary age children. They distinguished four categories of addition and subtraction problems: change, combine, compare and equalize (see Carpenter & Moser, 1982, 1984).

Carpenter and Moser (1984) conducted a research on primary school children to analyse their solution strategies according to the type of problem presented. The authors argue that the processes that children use to solve addition and subtraction problems are intrinsically related to the structure of the problem. This idea that addition and subtraction word problems differ both in semantic relations used to describe a particular problem situation and in the identity of the quantity that is left unknown is also supported by other researchers (see De Corte & Verschaffel, 1987; Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983; Fuson & Willis, 1986), who argue that addition and subtraction problem types are related to fairly systematic differences in children’s performance at various grade levels.

According to Nunes et al. (2005), children’s ability to solve problems involving an additive structure develops in three phases: first children can solve simple problems; then they can solve the inverse problems; and finally they can solve static problems. The addition and subtractions simple problems are those in which children are asked to transform one quantity by adding to it or subtracting from it (e.g., Joe had 5 marbles. Then he gave 3 to Tom. How many marbles does he have now?). These types of problems involve relations between the whole and its parts. The inverse problems are those in which the situation presented in the problem relates to a schema, but the correct resolution demands the inverse schema. For example, in the problem "Joe had some marbles. Then he won 2 more marbles in a game. Now Joe has 6 marbles. How many marbles did Joe have in the beginning?" (Nunes & Bryant, 2010a), subtraction appears as the inverse of addition; the quantity increased and the final one are given, and the initial quantity is unknown. The addition and subtraction static problems are those in which children are asked to quantify comparisons. For example, "Joe has 8 marbles and Tom has 5. Who has more marbles? (an easy question) How many more marbles does Joe have than Tom?" (a difficult question) (Nunes & Bryant, 1996; Nunes et al., 2005).

For Nunes and Bryant (1996) the difficulty of the problem is determined not only by the situation but also by the invariants of addition and subtraction that have to be understood by the children in order to solve a particular problem, and these invariants change according to the unknown parts of the problem. Nunes and Bryant (1996) also point out that the success in addition and subtraction tasks for young children is also determined by the resources that children are using to implement computational procedures, the system of signs. For the authors problems that involve relations are more difficult than those that involve quantities. The literature about additive reasoning has been giving evidence that compare problems, which involve relations between quantities, are more difficult than those that involve combining sets or transformations. Carpenter and Moser (1984) refer that many children do not seem to know what to do when asked to solve a compare problem.

Nunes et al. (2005) conducted a research with primary school Brazilian children, from grades 1 to 4, to analyse their performance when solving problems of additive reasoning. Their results indicate levels of success above 70% for the children of all grades when solving simple problems of part-whole relations involving addition and subtraction. When children were asked to solve inverse problems only 60% of the first graders and more than 80% of the 4^{th}-graders succeeded in a problem such as: "Kate had some candies. She won 2 more in a game. Now she has 12 candies. How many candies did Kate have in the beginning?". Their study also analysed comparative problems, such as: "In a classroom there are 9 pupils and 6 chairs. Are there more chairs or pupils? How many pupils are there more?". The authors reported around 50% of success for the second question, and almost 90% among the 4^{th}-graders. These results support the idea that the development of children's additive reasoning is progressive, but also suggest that children are able to solve many of these problems before they receive any formal instruction on addition and subtraction.

Literature gives evidence that kindergarten children are able to solve some addition and subtraction problems (see Fuson, 1992; Nunes & Bryant, 1996), but that does not mean that they understand all the relations in the context of additive reasoning problems. The children's understanding of addition a subtraction is progressive and develops over a long period of time.

To understand more about the children's additive reasoning, it becomes relevant to analyze younger children's ideas of addition and subtraction. Following previous research of Nunes et al. (2005), it was conducted a study with young children, from 4 to 6 years of age, concerning these issues. The study was developed to examine children's understanding of additive reasoning problems. For that two questions were addressed: a) how do children perform when solving additive reasoning problems?; and b) what explanations do they present when solving these problems?

METHODS

Individual interviews were conducted to 45 kindergarten children (4- to 6-year-olds), from Viseu, Portugal. There were 15 children from each age level. In these interviews children were challenged to solve 12 additive reasoning problems (4 direct problems, 4 inverse problems, 4 comparative problems). The interviews were conducted always by the same researcher.

The problems presented to the children were an adaptation of the problems previously documented in the literature by Nunes et al. (2005). Table 1 gives some examples of additive problems presented to children.

Type of problem	Example
Direct	Kate's mum gave her 4 pencils. Later she gave her 2 more. How many pencils does she have now?
	Ben had 7 candies and he gave 5 to his sister. How many candies does he have now?
Inverse	Anna had some candies. She gave 3 to her sister. Anna has 2 candies now. How many candies did she have in the beginning?
	Mark had 5 chocolate candies, he ate some and now he has 3 candies. How many chocolate drops did he eat?
Comparative	In a classroom there are 6 pupils and 4 chairs. Are there more pupils or chairs? How many more?
	Mary has 3 flowers. She has 2 more flowers than Betty. How many flowers does Betty have?

Table 1: Examples of additive reasoning problems.

All the problems were presented to the children by the means of a story problem and material was available to represent the problems.

No feedback was given to any child when solving the problems. All the children were asked "Why do you think so?" after his/her resolution in order to know children's arguments. In the comparative problems, it was expected that some children could requested help to understand the problem. In some cases the interviewer had to repeat the problem to the child or to put a second question, transforming a static question into a dynamic one, in order to facilitate their understanding of the problem. For example, instead of "how many cars are there more than planes?" – a static question – the child would then be asked "How many planes should we give to Mark for him to have as many toys has Ben?" – a dynamic question.

For all these problems, the assessment of children's performance was 0 for an incorrect response, and 1 for a correct one.

Data collection took place by means of video record and interviewer's field notes.

Results

A descriptive analysis of children's performance when solving additive reasoning problems was conducted. Table 2 summarizes this information for each type of additive structure problem according to the age level.

Additive reasoning problems			
		Mean (s.d.)	
Type of problem	4-year-olds (n=15)	5-year-olds (n=15)	6-year-olds (n=15)
Direct	2.13 (1.25)	3.75 (1.36)	3.53 (0.83)
Inverse	1.47 (1.30)	1.80 (1.27)	2.53 (1.25)
Comparative	0.80 (0.78)	2.33 (1.23)	2.33 (1.29)

Table 2: Mean and (standard deviation) of correct responses when solving the additive structure problems by age level.

It is remarkable the children's success levels when solving additive reasoning problems. Even the 4-year-olds were able to solve successfully some of these problems. The inverse problems and the comparative problems seemed to be more difficult for children than the direct ones, but even in those 5- and 6-year-olds children presented a correct resolution. The comparative problems were the most difficult for the children. Very often the interviewer had to repeat the problem to the child or to ask a second question in the same problem in order to facilitate children’s understanding of the problem, moving from a static question to a dynamic one, as referred before. Thus, the number of cases in which the interviewer had to transform a static problem into a dynamic one was registered producing two categories: without transformation, in which the child solved the problem with no changes; and with transformation in which the child need the interviewer to transform the problem. In any of these cases, the assessment was 0/1 for incorrect/correct responses.

Table 3 summarizes the number of correct responses given by the children when solving the comparative problems according to the need of changes in the presentation of the problem. As each child solved 4 comparative problems, 60 resolutions for each age group were produced.

	Correct responses in comparative problems		
Difficulty level	4-year-olds (n=15)	5-year-olds (n=15)	6-year-olds (n=15)
Without Transformation	2	14	19
With Transformation	10	21	16
Total correct responses	12	35	35

Table 3: Number of correct resolutions in the comparative problems, with the transformation and without it, according to the age.

Figures 1 to 3 present the distributions of the total of correct responses for the three types of additive reasoning problems, according to the age level.

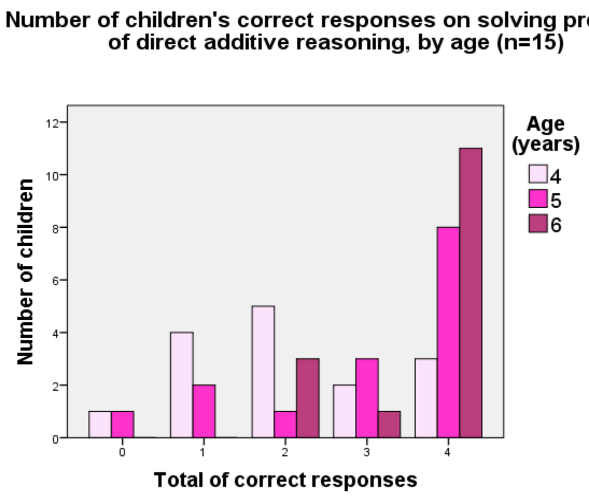

Figure 1: Number of correct responses for direct problems by age level.

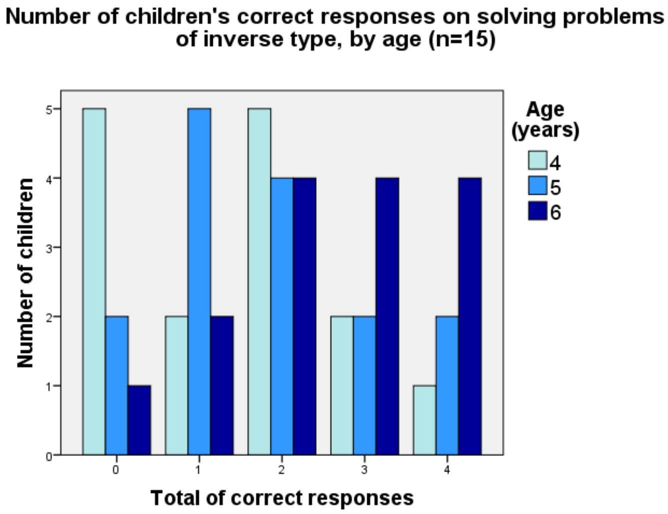

Figure 2: Number of correct responses for inverse problems by age level.

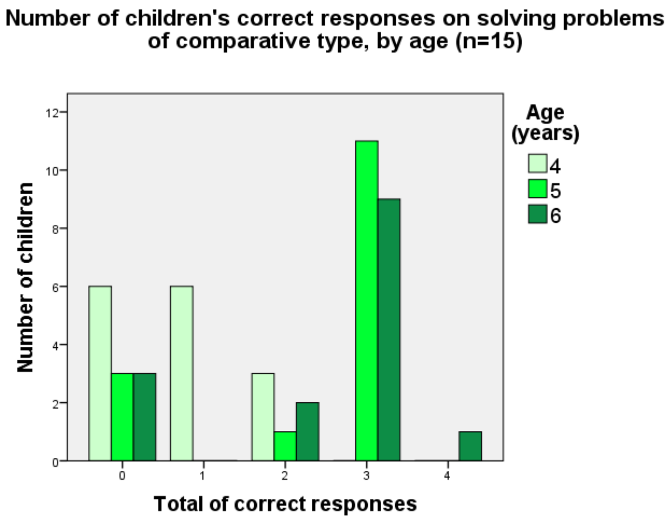

Figure 3: Number of correct responses for comparative problems by age level.

In order to analyse the effect of the age on children's performance solving the different types of additive problems a one-way Analysis of Variance (ANOVA) was conducted with performance in the type of problem (direct, inverse, comparative) as dependent list and age (4-, 5- and 6-year-olds) as a factor. There were no significant effects of the age on the direct problems neither on the inverse problems, but there is a significant effect of age on comparative problems ($F(2,42)=9.3$, $p< .001$) indicating that older children performed on this problems than the 4-year-olds. Bonferroni post-hoc tests indicate that children of 5- and 6-year-olds performed better than the 4-year-olds, but no significant differences were found on children's performance of 5- and 6-year-olds. Thus, in direct and inverse type of problems there was no age effect; the comparative problems were easier for older children than for the younger ones.

To know more about children's reasoning when solving these problems, their arguments were analysed for each type of problem. Four categories of children's arguments were considered in this analysis. The valid arguments comprise the justifications in which children consider all the quantities involved in the problem correctly; the incomplete category comprises children's arguments that refers only to one part of the quantities involved in the problem; the invalid arguments are those in which children do not articulate the quantities involved in the problems; and the no argument category that comprises all the cases of absence of argument.

Table 4 presents the number of arguments of each type that were used by children when solving additive reasoning problems correctly, according to the age.

Additive reasoning problems									
	Type of problem								
	direct			inverse			comparative		
Type of argument	4yrs	5yrs	6yrs	4 yrs	5yrs	6yrs	4 yrs	5yrs	6yrs
Valid	17	19	38	12	17	28	8	22	22
Incomplete	1	9	-	-	2	1	-	-	6
Invalid	3	8	4	7	2	7	3	9	4
No argument	11	9	11	3	6	2	1	4	3
Total correct resp.	32	45	53	22	27	38	12	35	35

Table 4: Number of arguments of each type given when solving the additive structure problems by age level.

Four categories of children's arguments were considered in this analysis. The valid arguments comprise the justifications in which children consider all the quantities involved in the problem correctly; the incomplete category comprises children's arguments that refers only to one part of the quantities involved in the problem; the invalid arguments are those in which children do not articulate the quantities involved in the problems; and the no argument category that comprises all the cases of absence of argument. Table 4 presents the number of arguments of each type that were used by children when solving additive reasoning problems correctly, according to the age.

Children of all age levels presented valid arguments were associated to correct resolutions. This suggests that the results obtained from children's performance are associated to an understanding of the problems presented to them. Around 53% of the 4-year-olds could solve correctly the simple problems presenting valid justifications; these percentage increases to almost 72% for the group of 6-year-olds children. Valid arguments were also presented in 54.5% of the correct answers given by the 4-year-olds children when solving the inverse problems, and in 66.7% of the correct resolutions of the comparative problems. In all type of problems there were children who were able to solve them correctly, but were unable to present a valid argument.

The use of an incomplete argument can be understood as child difficulty to articulate verbally a logic explanation that was carried on. Also children who solved correctly the problems presented no argument, as it happen with 34.4% of the 4-year-olds that solved correctly the simple problems.

DISCUSSION AND CONCLUSION

Children's informal knowledge is supposed to be the starting point for the formal instruction. Thus, it makes sense to know better what do children can and cannot do before being taught about arithmetic operations in primary school. The results presented here suggest that Portuguese kindergarten children are able to solve some problems involving additive structures with understanding, in particular conditions.

These results converge with those presented by Nunes et al. (2005) who analysed 5-8-year-olds children's performance when solving additive reasoning problems. These authors also reported that additive comparative problems were more difficult to young children than the direct and inverse ones. Our study extended these findings about children's additive reasoning as it gives evidence that 4-year-olds children can succeed in solving direct, inverse and also comparative problems. Their procedures do not vary from those used by the 5- and 6-year-olds relying on the schema of the act of join and separate for the direct and inverse problems previously identified in the literature (see Nunes & Bryant, 1996; Nunes et al., 2005).

The children's arguments were also analysed in order to get an insight on their reasoning when solving the additive structure problems. These arguments give evidence that children as young as 4 years of age can establish a correct reasoning and solve this type of problems. This suggests that their correct answers were not achieved by chance. If there are children of 4-year-olds able to solve some additive structure problems with understanding, relying in their informal knowledge, perhaps kindergarten could stimulate their early ideas about addition and subtraction. More research is needed to analyse these issues and to find out what sort of problems, if there are any, should be presented to kindergarten children in order to help them to develop their reasoning.

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DESIGNING TALES FOR INTRODUCING THE MULTIPLICATIVE STRUCTURE AT KINDERGARTEN

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We present a design study to introduce multiplicative thinking at Kindergarten level with an algebraic perspective. Starting from some theoretical assumptions about the psychological roots of multiplication and about the use of narration in Math Education, we build a suitable narrative context in order to promote children's actions consistent with such roots. We analyze the development of this path and its management, emphasizing the special role played by the dialectics between actions upon objects and graphic representations.

INTRODUCTION

The discovery in human beings of very early, if not innate, mathematical competencies, achieved by recent neuroscientific studies, induce to deepen the study of cognitive strategies recognisable as roots of mathematical structures and procedures, and to design learning environments to drive their evolution. This enterprise is not new, as it can be traced back to Piaget's studies about action schemata, from which a wide literature, in particular about the origins of arithmetical structures, has been produced. The common starting point is that action is at the root of any abstract thinking and in particular of the comprehension of arithmetical structures. This idea has been developed within different perspectives, also due to the increasing information we are gaining in these last years about our brain functioning (see e. g. Gallese & Lakoff, 2005).

In this field, our research group has been working for several years at the design and development of prototypes of long-term paths for primary schools aimed to promote in pupils arithmetical competencies as well as linguistic ones, in order to express and communicate their achievements. We are aware of the basic difference between actions upon objects and mathematical operations, but also of the neurophysiology discovery that the same neural circuits are deputed both to actions and to abstract thinking, therefore we think that to carefully identify the action schemata is fundamental in order to exploit them as roots: since these actions and the related mathematical operations will constitute the true base for the whole disciplinary structure.

In this paper we present a design study realized by our team in collaboration with an expert teacher: where a path is developed to introduce multiplicative thinking at a Kindergarten level with an algebraic perspective. A suitable narrative context was created in order to induce actions consistent with the theoretical roots of multiplication, identified according with some theoretical assumptions. We present these references in the next section, after which we briefly clarify the methodological equipment that has informed our experimentation; then, in the widest section we describe and analyze the main parts of the experimental path, and finally we draw some conclusive remarks from our research experience.

THEORETICAL BACKGROUND

In the last decades many studies have been developed about the cognitive roots of arithmetical structures. Without pretending to be exhaustive, we can distinguish two trends: to look for a correspondence between a given arithmetical operation (or arithmetical structure, i.e. the operation with its inverse) and an action scheme, as in Piaget or in (Davydov, 1992); or to classify the different situations in which the use of the operations is needed (e.g. Vergnaud 1983, Greer 1992, Steffe & Cobb 1998). The second kind of studies seems very useful especially for detecting cognitive problems that might underly a given recurrent mistake, whereas the first approach is more fruitful for planning class activities, particularly when arithmetic is addressed since the very beginning in an algebraic perspective, as in our approach (Iannece *et al.*, 2010).

In particular, we refer to Davydov's suggestion (1992) that rather than viewing different correspondences between each mathematical operation and an action schema, links the whole multiplicative structure to a specific psychological need. In his vision, indeed, the psychological root of multiplication is identified in the change of measure unit, when some magnitude has to be measured:

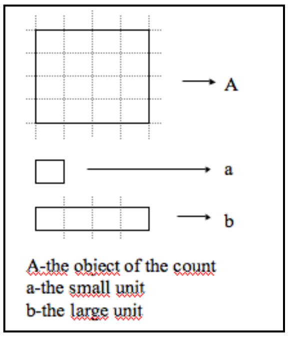

Figure 1

If the magnitude of an object is depicted by A, the small unit of count by a, the large unit by b, then the system of operation, carried out by determining the numerical value of A **indirectly** through a, can be expressed by the following formula: $\frac{b}{a}\times\frac{A}{b}=\frac{A}{a}$ (Davydov 1992, p. 11, see fig. 1).

According to Davydov's studies, we think that children can explore since kindergarten the arithmetical structures in an algebraic perspective by exploiting their cognitive strategies and using their languages. In this direction the graphic representation plays a special role since it can be viewed both as a perceptive metaphor of paradigmatic/structural aspects and as a cognitive support for generalization (Stetsenko, 1995). In Vygotsky's sociocultural vision of learning, in graphic representations sign and meaning arise together, then the integrated use of graphic, verbal and symbolic representations lets the concepts as well as the expressive tools develop. The functional role of drawing in children's cognitive and emotional development and its intertwinement with other communication tools have been explored in the sociocultural perspective. In particular it has been observed that

> young children do not radically differentiate between drawings and writing. At least part of this confusion must be due to the fact that children view both drawing and writing primarily as ways of communicating with others. (Stetsenko, 1995, p. 50)

In other words the intertwined development of drawing and of written and oral language in early childhood can be related to children's need to gradually grasp adults' means of communication. In this study we will show how this knot can be exploited and driven toward "paradigmatic" aspects of language, in particular by promoting the *array* as an effective representation of the multiplicative structure.

Our theoretical background includes also design-oriented studies about the role of tales to build mathematical meanings, in this case multiplicative ones. In the 1970s Donaldson has already observed how the child is particular sensitive to contexts where human intentionality can be recognized and how he uses this key to interpret and give meaning (Donaldson, 1978). To understand people's stories, reasons and feelings is linked to what Bruner calls "narrative" thought, juxtaposed to "paradigmatic" or "logic-scientific" thought. The complementarity of the two kinds of thoughts is put in evidence in several contexts of Math Education, as in problem solving activities (see e. g. Mellone & Grasso, 2008). About this, Zan (2011) observes how a word problem is both perceived as description of a 'human' situation, and analyzed for its paradigmatic features with the goal of solving a question. For this reason the mathematical information in a word problem has to be consistent with the narrated story and viceversa, in order to get resonance between the narrative thought and the paradigmatic one. Othewise the risk is to produce a

> 'narrative rupture' in the text of the problem, i.e. the question and the information needed for the solution are not consistent from the point of view of the narrated story. (Zan, 2011, p. 341)

As we will show in the sequel, we have tried to take this need into account in building the tale for our educational path, by describing characters who are moved by understandable feelings and goals, and by linking feelings and goals with the mathematical questions. Also the teacher's management of the activity has been careful in connecting and balancing the human and paradigmatic aspects of the story.

METHODOLOGY

The experience we are going to analyze comes from a wider research project carried out for several years in Naples by some researchers in Math Education and a group of Kindergarten, Primary School and Lower Secondary School teachers. This group has been working at building and validating prototypes of long-term paths for the teaching/learning of arithmetical structures in an algebraic frame. Common feature of these activities is the assumption of a Vygotskian perspective about learning, in particular on the role of signs in the semiotic mediation process. The research group has been working for several years about the use of the *array* as support for multiplicative thinking; in this study we explore the possibility of using such representation with 5-6 year-old children. To introduce multiplicative thinking in an algebraic perspective, we have built, in collaboration with a kindergarten teacher-researcher, a path that starts with the telling of a story. However, our goal was not just to validate in a class activity a path packed in advance, but rather to be able, starting from an initial plan, to repeatedly modify the path itself, according to classroom events and interactions, following in this a typical design study methodology (Cobb *et al.*, 2003). Consequently, the theoretical issues listed in the above section have not been transferred into action along a rigid sequence, but have been intertwined, in order to obtain effective outcomes for children.

In the next section we will illustrate the main parts of the design and of its three months development. Our collection of data includes children’s drawings, transcripts from class discussions, photos, audio and video recording.

THE TALE OF THE GLUTTONOUS KING AND THE DIDACTICAL PATH

The story that opens the path has been invented in order to merge a change of measure unit in a narrative context. The story tells the adventures of a king’s servant who has to do several trips through a tangled wood in order to reach a bakery and to buy cakes for the royal family, composed by four members. The cakes are carried ‘two at a time’ (first change of the measure unit) since the oven takes out only two cakes, one chocolate and one strawberry cake, each time, and each royal member wants to taste both. At the end of the story the teacher asks children to help the servant to pay the bill, knowing the total amount of the cakes bought (here, notice the care for consistency between narrative and paradigmatic aspects). As usual for the teacher, the story is enriched by every sort of details, concerning the different characters and the sequence of events; moreover the verbal language is accompanied by the mimic-gestural one, the exigence of a mime show and a dramatization naturally arise. In the first phase the tale is used to reflect upon the words meaning: for this purpose the children are invited to repeat the story and to discuss about the situation and the characters. The teacher suggests also to make a sort of proto-analysis of the text.

Afterwards the teacher asks children to represent the story with a drawing. In this way she wants to analyse which things have impressed more the children, in order to orient the didactic mediation toward the children’s needs and her goals. In this phase the children draw only the passages of the story that turn out to be more meaningful or simpler to be represented. The “paradigmatic” aspects are left apart, certainly also because the previous work about the characters has favoured the narrative thought (see e. g. fig. 2).

Figure 2

Figure 3

Figure 4

Figure 5

The teacher decides for a bodily work, as a premise for reflecting on actions, and also for reaching more paradigmatic representations useful for catching the mathematical meanings of the story. After all, if we recognize action schemata at the roots of comprehension, then we have to make actions. A motoric activity is organized to reproduce the path covered by the servant from castle to bakery: six traffic cones and a cloth tunnel represent the wood, so a gymkhana has to be made to reach the bakery (the class kitchenette), that contains two tiles as the cakes (fig. 3-5). Each child performs his/her own servant's path in order to interiorize the trip as a meaningful experience. This means to carry a plate, to reach the bakery and to buy two cakes, one chocolate and one strawberry cake, as many times as needed to satisfy all family members.

Finally, the children are invited to represent the trips made and the cakes taken each time. This time, all the children try to answer the numerical question: nobody feels inadequate, everybody is involved. This guarantees children's self-esteem and confirms the effectiveness of the teaching methodology employed, which includes a careful balance between the exigencies that all the pupils live successful experiences and that nontrivial disciplinary contents are addressed.

In children's drawings a major attention to the paradigmatic aspects of the story arises, maybe supported by the motory activity and, in particular, by the iteration of trips. In all the drawings we can "see" the multiplicative structure expressed by the grouping: the cakes are linked to the trips and drawn as rhythms of repeated plates (fig. 7), some children represent the trips as lines, (perhaps recalling the feature of the path, see fig. 6) or, in most cases, as half-circles, that recall the cloth tunnel. Only two children (one less than 5 years old) use a person-marker (fig. 9), while only Maria Giovanna outlines a sort of array (fig. 8). Ivana traces also the numerals 2 and 4, although as simple drawing ornaments (fig. 9). We have already observed in the theoretical section how fuzzy is the boundary between drawing and writing at this age, both abilities being linked to children's attempts to appropriate adults' means of communication.

Figure 6

Figure 7

Figure 8

Figure 9

The day after the teacher orchestrates a mathematical discussion about the different representations. This is a crucial part of the teaching mediation based on children's reflections upon their own and their fellows' behaviour. After the drawings of the previous day are distributed to the pupils, their comments rapidly focus on the effectiveness of the representations in order to share the best symbols used. Everyone illustrates the way he/she has represented trips and quantities, then everyone is invited to redraw his/her symbols on the blackboard. In this way all symbols are under the eyes of everybody, and thus, after an analysis and a comparison of their features, the children choose the most effective among them (fig. 10). The half-circle is selected as the best representative of a trip, against teacher's expectation, who hoped children would have chosen, since this phase, the array as a powerful sign to represent trips and quantities of cakes at the same time.

Figure 10

The next meeting between the teacher and the research group is devoted to understand why the children have not chosen the array, even though it appears in one of the initial representations (fig. 8), and why the collective discussion and the teacher's guide didn't induce this choice: maybe the two dimensions "trips" and "cakes at a trip" are not so meaningful till that moment, to deserve a special attention and a form of distinction. Therefore, and according to a Vygotskian approach, we decide to introduce an artefact, as a semiotic mediator for the two dimensions of the multiplicative structure: a rectangular tray divided in two-times-four boxes, into which the children can arrange the cakes during the dramatization. The teacher tells a further part of the tale of the Gluttonous King, in which the number of cakes for each trip is inverted with the number of trips, in order to suggest a two-dimensional representation, as well as to evoke a new change of measure unit (from "two at a time" to "four at a time"):[1]

> The Gluttonous King wants to organize a party for his family, where everybody will get a chocolate cake and a strawberry cake. Knowing that the baker has now a larger oven where four cakes at a time can be cooked, what has the servant to do?

Maria Giovanna: He must cut the cakes into small pieces.

[1] Some fragments of this activity have been already presented and discussed in (De Blasio, Grasso & Spadea, 2008).

Teacher: But the King doesn't like small pieces since he is gluttonous!

Martina: Otherwise they need a still larger oven.

Mattia: No, the servant must do several trips, carrying two cakes for every trip.

Teacher:: Look, I have prepared a tray for arranging the cakes. So, what has the servant to do?

Sara: He must go to the bakery, buy the cakes, and put them on the tray.

The above transcript shows how the teacher mediation tends to justify the resort to the artifact-tray. It is also interesting to notice how she refrains from directly intervening on Mattia's difficulty, who has not caught the change of measure unit from the first part of the story. Instead, she prefers to address the whole class, using a different strategy. Thus she encourages the children to a new dramatization, making the same path but using this time the special tray to carry the cakes. And when Mattia, at the end of his path, arranges the cakes grouping them by two, as in fig. 11, the teacher stops the play and lets all the children look at the tray.

Ciro: In this way it looks like the servant is gone twice and has got two cakes each time.

Mattia [resentful]: No, I went only once [Mattia changes the cakes arrangement on the tray, putting them in a unique row].

Figure 11

Figure 12

Figure 13

Obviously, what is really crucial is not the artifact-tray as itself, but the teacher's mediation that, by promoting a shared action schema, helps children to catch, from the collective discussion, the analogies and the differences between the two parts of the story. The social interaction works very well at this moment: Ciro's remark, which is in better accordance with the use of the tray, immediately produces Mattia's reaction. Teacher's suggestion to reason upon his actions and not only upon the narrated story turns out effective, indeed if Mattia gets angry for doing something different from what he thought (or for being misunderstood), from the other side he is ready to conform himself to the rules of the game. Finally the teacher invites a child to figure the cakes on the blackboard, promoting in this way another step from the representation of the experience through the object-tray toward a representation through signs on the blackboard. Martina goes to the blackboard and draws a first row with four circles, then she begins a second row, as in fig. 12. So, Martina's way of reporting the 'mathematical story' is a sort of rhythm, already implicit in some previous drawings, where the circles displayed in two rows clearly prefigure a typical array.

Teacher: Let's look at Martina's drawing. What does it suggest to you?

Mattia (and others): That he's gone two times… and has got four cakes.

Teacher: Do you agree that now we understand what the servant has done? [She takes two equal 'two times four' trays and puts them close to each other, but differently oriented, see fig. 13] What has changed?

Antonio: Now the oven is larger and cooks four cakes at a time.

Teacher: But is the number of cakes the only thing that's changed? How many times the servant comes from the bakery with his full tray to satisfy everybody?

M. Giovanna: Twice.

Chiara [pointing at the columns of the array]: One and two, one and two.

Mattia: I don't see any change!

Teacher: Are you sure? I see a difference….

Chiara [her hand traces a turning in the air]: They become equal just if we turn them.

Martina: Of course, since in this case the cakes are four and the times are two, while in the other case the trips were four and the cakes were two.

Ivana: But they are eight, anyway.

The use of the tray in the action simulation has well oriented Martina to appreciate the value of the array in representing the performed action. However, the teacher prefers to go back to the material representation via the tray, to promote an effective sinergy between syntactital and semantical aspects of the story. This helps the children to focus on what stays and what changes between the two situations, in order to discover the commutativity of multiplication. Moreover, teacher's pressing requests of precision stimulates a refinement of children's linguistic expressions, supported by reference to the concrete experience or, as well, by representation tools like the arrays. For example, for Martina it is important to drive attention to the concrete meaning of what they did, while Ivana's statement goes exactly in the direction of the multiplicative operation, overlooking the details of the two situations: anyway, they both obtain the same result of 8 cakes[2].

In the rest of the year the teacher has proposed many variants of the story, in which the numbers of trips and cakes varied, but with the usual care for the above discussed consistency between narrative and paradigmatic aspects. We have observed that not all the children used the array to represent the different situations. Our goal wasn't clearly to impose the array, that is to train them to adopt a mechanical automatism, rather our goal was, in Vygotskian words, to

[2] Similar behaviors have been observed in grade 3 children (see Mellone & Pezzia, 2008).

promote a “cultural” imitation, that is to drive children to repeat by their own a strategy after having experienced its effectiveness.

For this purpose, at a certain point the teacher decided to change the experience context and to work with rhythms of sounds. The children were invited to record the sound patterns, by recognising a group of notes repeated many times. As usual, they worked sharing, representing, and discussing. But this time the children naturally chose to represent each pattern of symbols, corresponding to a sequence of repeated sounds, one under the other instead of sideways, as in Martina’s ingenious drawing (fig. 12). In this way the children build an array, putting in evidence the role of multiplication and favouring an exploration of its properties. The possibility to experience the efficacy of the array in a new context allows children to recognize the structural analogy between two different situations. Finally, we can report that, during some further variations on the theme, children did make autonomous use of the array.

SOME CONCLUSIVE REMARKS

The design study presented above shows how the action schemata, evoked by telling a story in which the consistency between narrative and paradigmatic aspects is cared, can create resonance (in the sense of Iannece & Tortora, 2008) between children’s strategies and formal mathematical stuctures. In our opinion our study also confirms Davydov’s suggestion (1992) about the essential role played by the change of measure unit in giving sense to the multiplicative structure. Indeed, the story context allows to explore two semantical dimensions (trips and cakes for trip) and, at the same time, the peculiar syntactic properties of multiplication (as the commutative property). The dramatization lets the paradigmatical aspects arise and, on the other hand, the use of the *array* as a semiotic mediator leads the children to start using a genuine mathematical language to ‘put things in order’ (note the emerging of refined multiplicative expressions in Martina’s words “in this case the cakes are four and the times are two, while in the other case the trips were four and the cakes were two”). Finally, the analysis of the path shows a great difference between working with a representation proposed by others (the array at the beginning of the experience) and managing the same ‘linguistic’ tool autonomously (Chiara’s action on the array to recognize the commutative property). In this sense the adults’ cultural mediation in providing the array has to be very careful, due to the foreseeable children’s difficulties of interiorization.

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EXPLORING PARTITIVE DIVISION WITH YOUNG CHILDREN

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This paper focuses on a study with 4- and 5-year-olds children understanding of partitive division when discrete quantities are involved. The study analyse how young children understand the inverse divisor-quotient relationship when the dividend is the same. The participants were 30 kindergarten children from Braga, Portugal. Individual interviews were conducted when solving tasks involving the division of 12 and 24 discrete quantities by 2, 3 and 4 recipients. Results showed that 4- and 5-year-olds children have some ideas of division, can estimate for the quotient when the divisor varies and the dividend is constant, and can justify their answers. Educational implications of these results are discussed for kindergarten activities.

FRAMEWORK

Children learn a considerable amount about mathematical reasoning outside school known as informal knowledge. Literature refers that kindergarten children possess an informal knowledge relevant for many mathematical concepts (see Nunes, 1992; Nunes & Bryant, 1997). This informal knowledge should provide the building of formal mathematical concepts. Concerning the division, several authors suggest that young children can divide discrete quantities successfully (see Frydman & Bryant, 1998; Pepper & Hunting, 1998; Kornilaki & Nunes, 2005; Squire & Bryant, 2002), arguing that these children possess some type of informal knowledge related to the division of quantities, understanding the inverse relation between the divisor and the quotient when the dividend is the same.

Correa, Nunes and Bryant (1998) argue that sharing activities can be relevant in the understating of the inverse relation between the divisor and the quotient. Also Kornilaki and Nunes (2005) argue that understanding the sharing activity helps children to understand the logical relations involved in the division of quantities, i.e., the relation between the dividend, the divisor and the quotient.

When considering the division of discrete quantities it becomes relevant to distinguish the partitive and the quotitive division. In partitive division problem a set of objects is given to be divided among recipients, and the share that each recipient has received is the unknown part. (e.g., there is a set of 10 candies to be shared among 5 children. How many candies does each child get?). In a partitive division problem, the divisor is the number of recipients and the quotient is the share they receive. In quotitive division, there is an initial quantity to be share into a known number of parts. The size of the parts is the unknown (e.g., Mary has 12 candies and wants to give 3 candies to each of her friends. How many friends are receiving the candies?). In quotitive division problems, the divisor is the share to be given to each recipient and the quotient is the number of recipients. Concerning these types of divisions Kornilaki and Nunes (2005) argued that children understand more easily the partitive division than the quotitive division.

Research presents several results of young children procedures when solving division tasks involving discrete quantities (see Piaget & Szieminska, 1971; Desforges & Desforges, 1980; Frydman & Bryant, 1998; Squire & Bryant, 2002). Particularly, Correa, Nunes and Bryant (1998) when investigating the development of the concept of division in young children, examined whether children who could share would be able to understand the inverse divisor-quotient relationship in partitive division tasks when asked to judge the relative size of 2 shared sets. The participants were 20 children of 5-year-olds, 20 of 6-year-olds and 21 of 7-year-olds from Oxford, England. The authors investigated the children's understanding of the three-term quantity relationship in division when the dividend was constant and the divisor varies. In their experiment the experimenter shared a given amount (12 in some trials, 24 in others) of red and blue sweets between two groups of rabbits, one red and one blue, putting the sweets in the boxes attached to the rabbits' backs; the experimenter pointed to one blue rabbit and one red rabbit and each child was asked whether they had the same quantity of sweets or whether one of them received more sweets, and why did the child think so. The authors argued that "if the children succeed in tasks where the dividend is constant and the quotient is inversely related to the divisor, we can be confident that their success indicates some understanding of core relations in a division situations." (p. 322). Results showed that 9 of the 20 5-year-olds performed significantly above chance and about 30% were able to verbalize this inverse relation in their justifications and 11 out of 20 of the 6-year-olds scored above chance and verbalized the inverse relation between the divisor and the quotient in the partitive tasks. The authors also report age improvements between 5 and 7 years. Correa, Nunes and Bryant (1998) also analysed children's justifications according to children's age. Most of the 5-year-olds were not able to give a mathematical justification for their choices and did not mention facts relevant to the solution of the task. The 6-year-olds presented justifications that revealed a progress from some comprehension of sharing and numerical equivalence to the understanding of the inverse divisor-quotient relationship. The majority of the justifications presented by the 7-year-olds showed a logicomathematical approach, referring the inverse divisor-quotient relationship.

More recently, Kornilaki and Nunes (2005) investigated whether the children could transfer their understanding of logical relations from discrete to continuous quantities. Among other things, the authors analysed 32 five-year-olds, 32 six-year-olds and 32 seven-year-olds solving partitive division tasks involving discrete quantities. In this type of problems the number of recipients varied to produce two conditions: 1) in the same divisors condition, the size of the divisor was the same; 2) in the different divisors condition, the number of recipients varied. The results showed that the different divisors condition was clearly more difficult than the same divisors condition. Thus, the authors argued that the inverse relation between the divisor and the quotient is understood later than the equivalence principle of division. The authors also pointed out that in partitive division tasks, one-third of the 5- and 6-year-olds justified their responses as “the more recipients, the more they get”, but this response decreased markedly with age as only slightly more 10% of the 7-year-olds used this incorrect reasoning.

The studies of Correa, Nunes and Bryant (1998) and Kornilaki and Nunes (2005) give evidence that, at age of 6 and 7, children have an insight into relations between the division terms, long before they are introduced to this operation at school. If previous research reports some success with 5-year-olds children, how would children of 4-year-olds would perform? Besides, it becomes relevant to get a better insight on young Portuguese children’s informal knowledge of division.

This paper focuses on young Portuguese children understanding of division of discrete quantities, when solving partitive division problems. For that we tried to address three questions: 1) How do children estimate the quotient in a partitive division in which the divisor varies and the dividend is kept constant? 2) How do children perform the partitive division tasks involving discrete quantities? 3) What procedures do they use in this process?

METHODS

A study focused on young children’s ideas of partitive division was conducted to address these questions. The participants were 15 four-year-olds (11 boys and 4 girls, mean age 4 years and 6 months) and 15 five-year-olds (7 boys and 8 girls, mean age 5 years and 6 months) from Braga, Portugal.

The participants were interviewed individually by one of the researchers when solving the problems. Each problem was presented to each child using a story and manipulatives representing the items involved in each story were available.

Each child was presented to 6 problems: 3 involving the division of 12 units (carrots) by 2, 3 and 4 recipients (rabbits), respectively; and 3 problems involving the division of 24 units (cabbage) by 2, 3 and 4 recipients (rabbits).

In the interview, first children were invited to estimate the effects on the quotient of increasing the divisor keeping the dividend constant. Then they were asked why they thought so. The idea was to have an insight on children’s understanding of the inverse divisor-quotient relationship when the dividend is constant. Then children were asked to carry out the division. In this process, their ability to perform the division was assessed as well as the procedures used by them.

The story presented to the children involved a context in which a white little rabbit had 12 carrots. Then he had to share them fairly with his friend, the brown rabbit. At this moment the child was asked: "Do you think that the white rabbit would be with more or less carrots? Why?". Them the child was invited to accomplish the division between the two rabbits. Them the child was asked: "Do you think that both rabbits are happy with this division of the carrots? Why?", "How many carrots did each received?". Then a little grey rabbit came around and they had to put all the carrots together again and share them among the three rabbits. "Do you think that each rabbit is going to have more or fewer carrots now?"; "Can you help the rabbits to share the carrots?"; "Do you think that all the rabbits are happy with this division? Why?". The story continues to include the black rabbit. The same questions were asked. In the very end, when the last rabbit came, the children were asked: "Do you think that all the rabbits are happy with this division? Why? Do you want to check it by counting?".

When the 24 units were involved, an analogous story was presented to them but now involving the 2, 3 and 4 rabbits and 24 cabbages.

Each child took approximately 20 minutes to solve all the problems, in spite of having no limit for it.

RESULTS

In order to understand children's ability to estimate the quotient in a partitive division in which the divisor varies and the dividend is kept constant, their correct responses and justifications were analysed. Table 1 resumes the percentage of correct estimates and valid justifications for the division of 12 and 24 units, according to the age. A valid justification is an argument in which a child expresses some ideas of the inverse divisor-quotient relationship, such as "because there are more rabbits and each one get fewer carrots." or "they will have fewer carrots because now there is the X rabbit".

	4-year-olds		5-year-olds	
	Correct resp.	Valid argum.	Correct resp.	Valid argum.
12 units	67%	43%	72%	67%
24 units	71%	52%	78%	83%

Table 1: Percentage of correct responses and valid arguments when estimating for the quotient with the dividends of 12 and 24 units, respectively.

It is interesting to note that children's performance in the estimating tasks improved from the first part of the problems (involving 12 units) to the second one (involving 24 units), in spite of the sizes of the initial sets. Perhaps this is due to the fact that when the problems involving the 24 units were presented to the children, they were not a novelty anymore.

Another remarkable point is the success observed among the 4-year-olds when asked to estimate and justify their judgement. Almost half of the children presented a valid justification for their correct answer when dividing the 12 units; when they were asked to divide the 24 units, their valid justifications increased slightly above 50%. These results suggest that children of 4-year-olds may have some ideas about the inverse divisor-quotient relationship presented in these conditions.

Children performance was analysed solving division tasks involving 12 and 24 units by 2, 3 and 4 recipients, respectively. Tables 2 and 3 resume the percentage of children's correct responses by age level, in these problems.

	12 units	
	4-year-olds (n=15)	5-year-olds (n=15)
Division by 2	87%	87%
Division by 3	67%	80%
Division by 4	67%	80%

Table 2: Percentage of correct responses by age level when solving the division of 12 units by 2, 3 and 4 recipients.

	24 units	
	4-year-olds (n=15)	5-year-olds (n=15)
Division by 2	60%	80%
Division by 3	86%	74%
Division by 4	67%	80%

Table 3: Percentage of correct responses by age level when solving the division of 24 units by 2, 3 and 4 recipients.

The results suggest that for young children it becomes more difficult to accomplish the division of 24 units than the division of the 12 units set, possibly due to the magnitude of the set.

As the children's performance was not normally distributed a Mann-Whitney U Test was conducted in order to analyse children's performance dividing 12 and 24 units according to the age level. The results show no significant differences on children's performance when dividing 12 units according to the age levels (age 4, Mdn=3, age 5, Mdn=2, U=149, n.s.) and when dividing 24 units according to the age levels (age 4, Mdn=3, age 5, Mdn=3, U=128, n.s.). Thus, results give evidence that there is no difference of 4- and 5-year-old children's performance in this division tasks.

Trying to explain these results, children's procedures were analysed when dividing 12 and 24 units by 2, 3 and 4 recipients, respectively. The same procedures were observed when children were dividing 12 and 24 units. The procedure I comprises the sharing procedures relying on the correspondence one-to-one by the recipients; the procedure II comprises the counting procedures; procedure III comprises sharing activity based on perceptual influence ignoring the size of the shares; and procedure IV comprises sharing activity combined with counting to produce equal shares.

Tables 4 and 5 resume the observed procedures used by the children of both age groups when solving the division problems of 12 and 24 units, respectively.

	12 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
Type of procedure	I	II	III	IV	I	II	III	IV
Division by 2	10	0	3	2	8	2	1	4
Division by 3	9	0	5	1	8	2	3	2
Division by 4	9	1	3	2	8	2	4	1
Total (Max.=45)	28	1	11	5	24	6	8	7

Table 4: Children's procedures solving the division of 12 units, by age level.

	24 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
Type of procedure	I	II	III	IV	I	II	III	IV
Division by 2	7	0	6	2	9	2	4	0
Division by 3	9	0	5	1	6	2	4	3
Division by 4	9	1	4	1	6	3	4	2
Total (Max.=45)	25	1	15	4	21	7	12	5

Table 5: Children's procedures solving the division of 24 units, by age level.

The procedures used by children did not change much according to the magnitude of the set to divide. Tables 4 and 5 suggest that sharing assumes an important role on children's performance when solving division problems, with discrete quantities. The sharing activity developed by each child and the type of shares produced give us an insight of children's ideas of fare share. Many 4-year-olds children used sharing activity without recognizing the need of producing fare shares, either when 12 or 24 units were involved (24% and 33%, respectively). This phenomenon was also observed in some 5-years-old children when 12 and 24 units were involved (17.8% and 26.7%, respectively). Nevertheless, the majority of the children of both age groups involved in this study recognized the importance of producing fare shares in the division tasks presented to them.

The procedure mostly used by both age groups of children was correspondence one-to-one. This procedure conducted children to correct resolutions, producing fare shares. The procedures using sharing activity based on perceptual influence ignoring the size of the shares were also popular among children of both age groups.

After carry out the division of the items by the recipients, the children were asked if they were happy with the division made through the question "Do you think that all of the rabbits are happy with this division? Why?". They were also challenged to verify their results by counting - "Do you want to check it by counting?" - to deepen the understanding of children's ideas of fare sharing by giving them an opportunity to correct themselves. Their reactions were analysed and allowed us to distinguished the following categories: CcE comprises children's verifications in which it was observed Correct counting of the items in each recipient when there are already equal shares; CcNon-NE comprises children's verifications in which it was observed Correct counting of the items in each recipient, but without equal shares; NnC comprises children's reactions in which they refuse to verify because they are sure about it and it is correct; NvNE comprise their reactions in which they do not recognise the need to verify and equal shares were not produced; NC comprise children's reaction in which the correct counting of the items was not accomplished.

Tables 6 and 7 resume children's reactions, by age group, when solving the division tasks of 12 and 24 units, respectively. The majority of the children of both age groups used the opportunity to verify their shares, correcting their distributions when necessary. This was observed by 60% of the 4-year-olds and 73.3% of the 5-year-olds when 12 units were involved; and by 51.1% and 62.2% of the 4- and 5-year-olds, respectively, for the 24 units. These results suggest that equal share is a concept understood by young children of 4-year-olds. In most of the problems presented to them, these young children recognised the importance of fair shares when accomplishing a sharing activity in a division of discrete quantities.

	12 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
	Division				Division			
	by 2	by 3	by 4	Total	by 2	by 3	by 4	Total
CcE	9	10	8	27	11	12	11	33
CcNon-NE	2	3	4	9	3	1	3	6
NnC	0	0	0	0	1	1	1	3
NvNE	2	1	1	4	2	1	0	3
NC	2	1	2	5	0	0	0	0

Table 6: Children's reactions to the produced shares after dividing 12 units, by age level.

	24 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
	Division				Division			
	by 2	by 3	by 4	Total	by 2	by 3	by 4	Total
CcE	9	10	8	27	11	12	11	33
CcNon-NE	2	3	4	9	3	1	3	6
NnC	0	0	0	0	1	1	1	3
NvNE	2	1	1	4	2	1	0	3
NC	2	1	2	5	0	0	0	0

Table 7: Children's reactions to the produced shares after dividing 24 units, by age level.

It was also possible to observe a few children who did not need to verify their resolutions that were correct, being sure about their procedures and solutions obtained. A groups of children of both ages did not recognised the need of produce equal shares, in spite of using counting properly when verifying their results (20% and 13.3% of the 4- and 5-year-olds, respectively, when dividing 12 units; and 20% and 35.5% of the 4- and 5-year-olds, respectively, when dividing 24 units).

DISCUSSION AND CONCLUSIONS

The results presented here give some insights of young children ideas of division of discrete quantities but also their ideas of fair sharing. The findings of the study reported here suggest that young children of 4- and 5-year-olds possess some ideas related to the division of quantities, understanding the inverse relation between the divisor and the quotient when the dividend is the same. The analysis conducted here give evidence that children of 4-year-olds reveal some understanding of the effect of increasing the number of recipients when the amount to share is constant. These children were able to estimate the result of division. This suggests that children also have some ideas of the inverse divisor-quotient relationship in partitive division tasks, when asked to judge the relative size of shared sets. This idea is in agreement with Frydman and Bryant (1998), Correa, Nunes and Bryant (1998) and Kornilaki and Nunes (2005).

The study reported here has some similarities with some presented previously in the literature (see Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) but also offers some original contributions. Correa, Nunes and Bryant (1998) investigated 5- to 7-year-olds children's understanding of inverse divisor-quotient relationship, when partitive division was involved. Their findings give evidence that 5-year-olds children can succeed in these tasks. Also Kornilaki and Nunes (2005) give evidence of 5-year-olds children success when solving this type of tasks. In our study we analysed how children of 4- and 5-year-olds behave when dealing with this type of problems. Some positive signs arise from this investigation. Four-year-olds children are also able to understand some ideas of divisor-quotient relations in particular conditions.

The procedures used by the children of this study suggest that correspondence can play an important role on children's sharing activity and on their accomplishment of division. Some authors argue that sharing activities can be relevant in the understating of the inverse relation between the divisor and the quotient (see Correa, Nunes & Bryant, 1998) and that understanding the sharing activity helps children to understand the relation between the dividend, the divisor and the quotient (see Kornilaki & Nunes, 2005). In agreement with these ideas, one-to-one correspondence sustaining the sharing activity seems to allow young children to understand the logical relations involved in the division of quantities. This study also shows that equal share is a concept understood by some 4-yaer-olds children and recognized by them as an important issue of the division of discrete quantities. Nevertheless, fair sharing does not seem to be only concept for understanding the division of these quantities, as many young children were able to estimate the effects of increasing the divisor in the quotient, for the same dividend, before carry out the division.

These findings suggest that kindergarten activities could stimulate children's early ideas of division, relying of their informal knowledge. These activities could comprise the use of share and the production of equal shares, but also activities to promote the understanding of the logic relations involved in the division, when the dividend is kept constant. These ideas are crucial to understand some complex mathematical concepts such as fractions, later on in the formal traditional school.

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THE APPEARANCE OF EARLY GENERALIZATION IN A PLAY[3]

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The paper shows the appearance of generalization and its fundamental role in a didactical activity based on a play with rules, proposed to pupils 5-7 years old. Every play requires and promotes different competences, in particular logical and mathematical. The study of pupils' behaviours in front of the task furnishes some examples that prove the possibility of an early mathematical activity of generalization.

THEORETICAL FRAMEWORK

Usually the word 'generalization' is related to algebraic procedures and reasoning, but it is possible to observe the use of generalizations also in other mathematical activities. Generalization is often cited as typical form of mathematical thinking, but without using a definition or specify its meaning. Moreover generalization is often associated with abstraction, since the boundary between them is very thin.

In an Italian book for teachers, we can read this definition of 'generalization':

> the capability to *free oneself from particular*, to find *solutions more amply valid* to achieve a given aim. … capability that allows to distinguish the essential from the particular, "what it *needs* make in given situations" from the various "way in which it *can* be made". (Altieri Biagi & Speranza, 1981, p. 178)

In her analysis of the act of understanding, Sierpinska considers four basic mental operations: identification, discrimination, generalization and synthesis. Her definition of generalization is the following that completes the previous:

> Generalization is understood here as that operation of the mind in which a given situation (which is the object of understanding) is thought as a particular case of another situation. The term 'situation' is used here in a broad sense, from a class of objects (material or mental) to a class of events (phenomena) to problems, theorems or statements and theories. (Sierpinska, 1994, p. 58)

In his theory of 'universal model' Hejny (2004) distinguishes six different stages: motivation, isolated (mental) models, generalisation, universal (mental) model(s), abstraction, abstract knowledge. In particular, concerning the 'Stage of generalisation' he writes:

[3]Work done in the sphere of Italian National Research Project Prin 2008PBBWNT at the Local Research Unit into Mathematics Education, Parma University, Italy.

> The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme generalising all these models is discovered. The stage of generalisation does not change the level of the abstraction of thinking. (Hejny, 2004, p.2)

Hejny (2004) writes also:

> The generalisation of isolated models (experiences and pieces of knowledge) is determined by finding connections between some of isolated models. This web is the most important product of the stage of the isolated models. (Hejny, 2004, p.5)

In this paper the author presents and studies an example of generalization that appears during a play. It is well known that the play can promote logical and mathematical competences. Schuler (2011, p. 1912) highlights that:

> […] *play* and *relationship of playing and learning* have to be explored more closely when talking about mathematics for the early years.

Starting from the consideration of emotional, social and cognitive role of the play, she writes:

> […] play in early childhood is the motor of development and hence associated with learning. Consequently the underlying question seems not to be "Can children learn while playing?" but rather "How can learning while playing be modeled?" and "Can children learn mathematics while playing? (Schuler, 2011, p. 1913)

After an analysis of some theoretical models, she emphases "the central role of the educator and the quality of materials, games and activities". In fact, sometimes it is difficult to adopt a good equilibrium between a free and spontaneous play and a guided play. In other words, "Play is not enough. […] children need adult guidance to reach their full potential" (Balfanz et al., 2003), but when the teacher proposes a play finalised to promote particular abilities, he risks to force in some way the child and to impose directions of work connected with the play finality. In particular, Schuler (2011) studied situational conditions of learning while playing and she highlights three main blocks: affordance, liability and conversational management:

> […] rules can offer mathematical activities beyond a material's intuitive affordance and thus create liability. Intuitive affordance of materials is replaced in games by (*the affordance of*) keeping the rules and winning *the game*. (Schuler, 2011, p. 1919)

In the play utilised in the present research, an important role is done to row-column arrangements. Rożek & Urbanska (1999) studied in depth this topic:

> The children have a different awareness of the rows and columns arrangement. Some of them prefer rows, some of them columns. It appears that it was difficult to see both rows and columns, especially for young children.

In particular, Rożek in her researches about SCFL (Series-Columns Figures Layout) Rożek (1997, 1998) analyses children's behaviours, in terms of two activities constructing and drawing SCFL. She studies also verbal descriptions of SCFL and she organises the protocols in base of three different features: following the features of structures, following visual perception, using language. In the first, she observes the distinction between geometrical aspects as rows and columns or numerical aspects. In the second, she classifies the vision as global or analytical. In the third, the focus is on the language that can be referred to real world or in comparison with mathematical language. In our research, there is a part related to 'construction' and a second part based on 'lecture' of villages (2D) or palaces (3D), that can be analysed and organized following Rożek theory.

RESEARCH QUESTIONS

The present research is placed in the theoretical framework of early mathematical education by play, in particular it deals with children's development of reasoning in playing with rules. The initial hypothesis is that a suitable play can promote an early and spontaneous use of generalization. Our aim is to give answers to the following questions:

1. Is it possible to develop in children the construction of metacognitive instrument of generalization in the context of a guided play?
2. Under what conditions we can obtain learning of generalization, using a game that can promote it?

THE EXPERIMENT AND ITS METHODOLOGY

In this paper we present a research focused on a part of a wider study based on a play with rules, the 'Play of coloured houses', showed and analysed in a working seminar presented from the author in a CME conference (Vighi, 2010b). The main research aims were to study spontaneous reasoning made from children, playing with 'the play of coloured houses', to analyse their behaviors in front of row-column arrangements tasks and the possible recourse to metacognitive processes of symbolization and formalization. In this paper we refer only the part related to the appearance of generalization during the play and its crucial role. The experiment took place in the last year of kindergarten in which pupils (5-6 years old) worked in groups of seven or eight and in the first year of primary school (pupils 6-7 years old) with work in pairs. Pupils involved were 20 in kindergarten and 26 in primary school. The activities took place in every day context. In kindergarten they were conducted from the teacher[4] in presence of a researcher (the author of the present paper). Teacher presented the play and she conducted the works, promoting and fostering the viewpoints of

[4] I wish to thank the teacher Palma Rosa Micheli (Scuola dell'Infanzia Statale "Lodesana", Fidenza (PR), Italy), for her collaboration and helpfulness.

children, without force their thinking, but waiting to listen their ideas and observing their behaviours. Researcher observed, recording on video, later she analyzed and transcribed dialogues, making also written observations. In primary school the activities were conducted in part from the teacher[5] and in part from the author who worked with children in pairs.

THE "PLAY OF COLOURED HOUSES"

The "Play of coloured houses" is a play without winner, based on a disposition of houses with three different colours (red, yellow, green) in a grid 3x3, respecting the following rule: in each row and in each column it needs to have houses of three different colours. We report here some examples:

The play remembers Sudoku, in fact it can be seen as a simplified version of Sudoku with a grid 3x3 (instead of 9x9) and only three 'symbols' (it is possible to use digits 1, 2, 3 in place of colours). From the mathematical point of view, it is a 'Latin Square', i. e. a square in which "each element appears only one time in each column and only one time in each row" (Quattrocchi, Pellegrino, 1980).

The play requires the contemporaneous management of rows, columns and colours. It can be executed by means of 'method of attempts and errors' or using rules discovered during the play: "It is impossible to have a red house here", or "Here it must be a yellow house" etc. When a pupil plays, he makes argumentations, and also hypothetic-deductive reasoning: "If I put here a green house, then …" and so on.

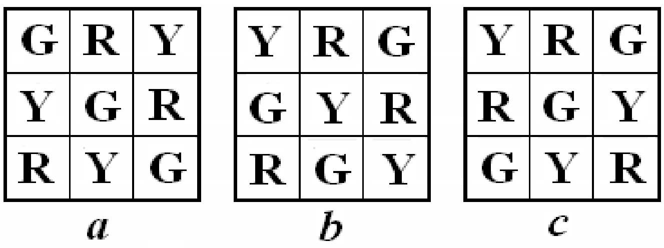

Figure 1: Examples of villages

THE 'SCALETTA THEOREM'

In scholastic year 2009/10 the "Play of coloured houses" was presented in kindergarten in a context of motor activity, after pupils played with coloured tiles and a support for tiles organized in three rows and three columns (Fig. 2). We drew a house on each tile with the aim to give an orientation that allows to distinguish clearly the built villages (in this way it is possible to have 12 different villages).

[5] I wish to thank also the teacher Ines Tommasini (Scuola Primaria Vicofertile (PR), Italy).

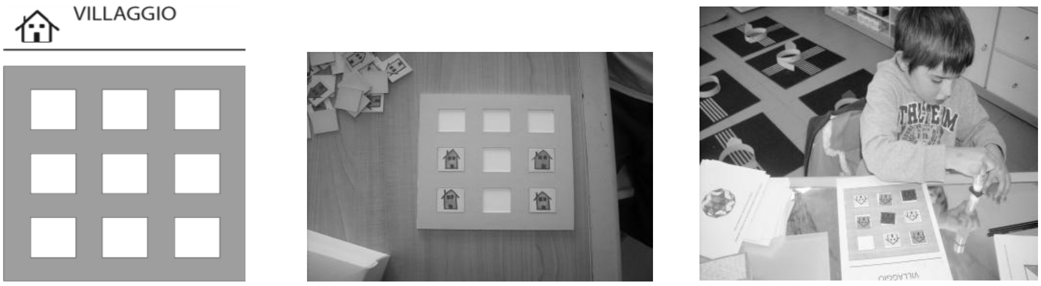

Figure 2

In scholastic year 2011/12 we presented the same activity in Primary School (pupils 6-7 years old). Here we refer only on comparison of villages constructed from pupils, suggested from the teacher. It is well known that the activity of comparison is fundamental in mathematics, to construct concepts: thinking about analogies and differences can promote the formation of a concept. It is also documented that comparison it is not spontaneous in young children; they start using intuition, but it is insufficient, so it compels the use of the language. After a lot of activities based on the play, teacher submitted couples of villages and she solicited their comparison starting from a couple of ‘equal villages’, and continuing with couples of villages with ‘the same structure’ etc. An important observation is about the different ways of seeing the SCFL (Rożek, 1997) that children showed: use of a local way of seeing, observing only some couples of tiles with the same colours, placed in the same places (“In the first village there is a green house here, in the second also”); observation of the disposition of all the tiles with the same colour and use of a words of natural language to describe their disposition (“It seems letter C”); recognition of rhythms or cycles (“red, yellow, green, red, yellow, green, …”); individuation of symmetric villages (for instance, villages *a* and *c* in Fig. 1); only observation of rows (or columns) and their exchanges (in Fig. 1, “The second row in village *b* is equal to third row in village *c* and vice versa”); observation of different orientations of diagonals (in Fig. 1, referring to *a* and *c* villages: “ … but one go down, the other go up”); description of features of diagonals (“In one diagonal there is the same colour” and “in the other diagonal there are three different colours”).

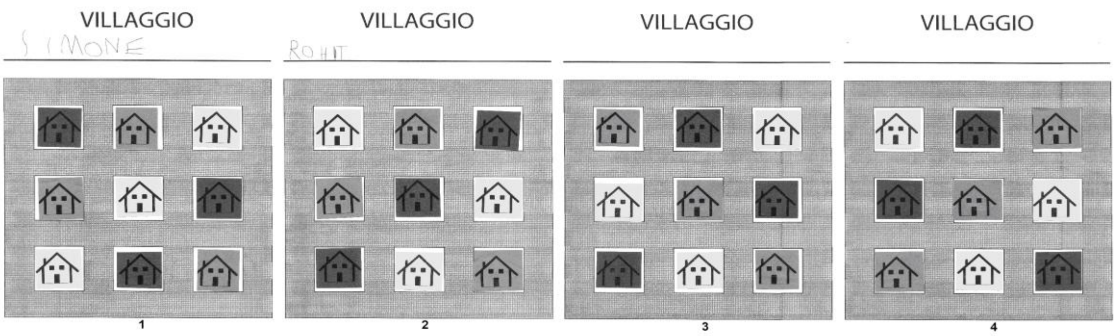

Figure 3

This last aspect suggested to the author of the present paper to put attention and to focus this topic: the visual perception of colour leads some pupils to move their attention from rows and columns, explicitly mentioned from the rules of the play, to diagonals that present a particularity, all tiles have the same colour. Our hypothesis is that it could be a starting point to investigate if children use or not ‘diagonal rule’ to make generalizations.

In the first experimentation, pupils of kindergarten school used the name ‘scaletta’ (in Italian language it means “little ladder”) to indicate this monochromatic diagonal; in fact, the disposition of tiles suggested the steps of a small ladder. The observation of ‘scaletta’ was developed in the following context: firstly each pupil constructed his village, gluing tiles on a sheet of paper expressly prepared for the use (Fig. 2); in a second moment teacher put some villages on a wall of the classroom and she asked observations from the pupils. In particular, they told: “The yellows are in single line” and “They are in angle”, “They are in little ladder”, “In a bandy row” (diagonal), “In a bandy row there are three equal colors, in the other bandy row there are three different colors”. It happens since teacher promoted the passage from micro-space to meso-space (Brousseau, 1983): micro-space is near to the subject and accessible to manipulation and vision, meso-space is accessible to a global and simultaneous vision (macro-space is accessible only for local visions). In fact, the first work proposed to the pupils took place in the space of the desk (micro-space), the second in the space of the classroom (meso-space). It changed the point of view in village’s observation: from rows and columns to diagonals. So, the “connection between some of isolated models” (Hejny, 2004) creates a web that produced generalization.

So, we observed an unexpected fact: pupils found and formulated a theorem that is a consequence of the play’s rule. We call it, the “Theorem of little ladder”: “In all villages there is a little ladder with only one colour”. It is an example of generalization in the meaning of Altieri Biagi & Speranza (1981): from particular to the essential.

Sometimes pupils used this theorem in their following constructions of villages that started from a diagonal monochromatic. In this way they adopted a strategy of village’s construction that involved new rules, different from these suggested from the play. It is a generalization as ‘capability to find solutions more amply valid’ (Altieri Biagi & Speranza, 1981), and also in sense of ‘a given situation is thought as a particular case of another situation’ (Sierpinska, 1994), but also in which the structure appears as generalizing isolated models in sense of Hejny (2004). But … the use of the theorem doesn’t guarantee success. It is evident in Chiara strategy (Fig. 4).

Y			Y	R		Y	R	G	Y	R	G	Y	R	G	Y	R	G	Y	R	G
	Y			Y			Y			Y		R	Y		R	Y	R	R	Y	R
		Y			Y			Y	G		Y	G		Y	G		Y	G	G	Y

Figure 4

Chiara started with a yellow diagonal, she continued with two correct passages, after she makes an error that leads to have at the end a 'wrong village'.

PASSAGE FROM 2D TO 3D PLAY

In the present school year, we decided to submit to the pupils of kindergarten (5-6 years old) a new version of the play, in three dimensions: it consists in the construction of a 'palace' of three floors (a cube 3x3x3), with similar rules: "In each wall face it needs to have three different colours in each row and in each column"[6]. The play can be considered a three-dimensional (3D) version of the two-dimensional (2D) play of coloured houses. Sometimes in mathematics we observe the use of the locution 'generalization' also for the passage from 2D to 3D.

Pupils worked in groups following the indications suggested from the teacher. She arrived in classroom with two big boxes and she created a condition of waiting about their contents. After, slowly she opened the boxes extracting cubes (27 wooden coloured cubes, 9 red, 9 yellow, 9 blue), their wooden support (Fig. 5), named from children "palace" or "house with a lot of floors", and a wooden rotating disk to facilitate gestures and the observation.

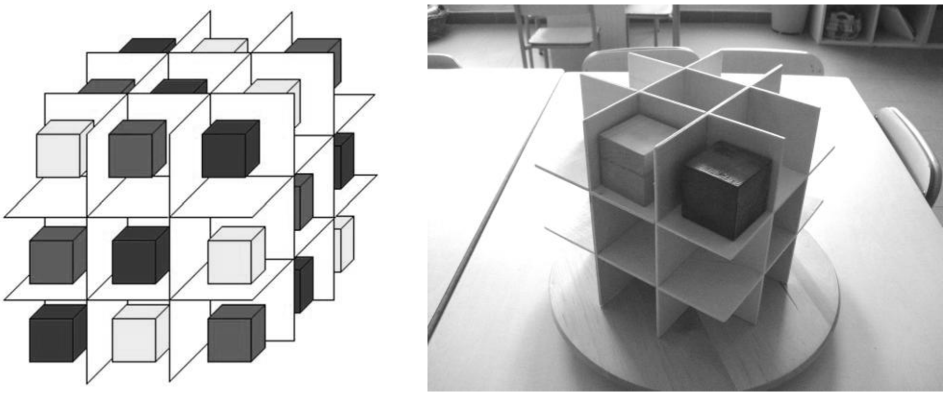

Figure 5

Firstly teacher suggested different free plays with cubes, after she invited each child to put a coloured cube on the support, promoting the construction of a building respecting rules; in a second time, she removed the support and she putted the cubes one near to the other (Fig. 6).

This choice promoted an important breakthrough, since, as Rożek (1997) write, in a row-column arrangement of figures the distance between objects influence in depth the observation. We choose to report here the development of the work in a group, named G2, but we could observe similar behaviours in other groups, of course not in all. In G2, a child observed the yellow diagonal present in the "roof of the palace" (Fig. 6), suggested from the colour and also from idea of "straight line" and he said that there was a mistake in the constructed palace.

6 A similar problem was studied from M. Gardner (1980) that found only one solution for the final cube (excluding rotations, reflections or permutations of colour).

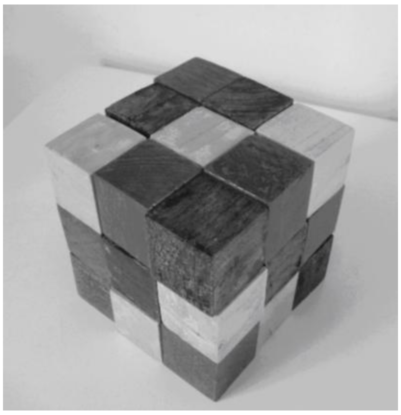

Figure 6

Teacher suggested that in fact all rows and columns respected rules and the child replies that “Yellow cubes are in point, as point of knife”. Immediately pupils find ‘points’ (‘scalette’ in the previous experience) in the other faces of the cube: “There are three points blue and three points red”. In fact, after the construction it is possible verify that the rule is respected also in the ‘horizontal floors’: in a floor there is a diagonal red, in another blue, in another yellow. So, they conclude that “This cube is magic!”.

Another breakthrough happens when a child observed that the other diagonal on the roof presented three different colours: he indicates it with his hand accompanying with gesture and sound: “here, blue, red, yellow, pum, pum, pum” and he repeated it for each face visible of the cube. He added: “A ‘point’ entirely yellow, another of three colours, it is an X”. We name it the “Theorem of two diagonals”. In other words, the disposition of diagonals in each face of the cube suggested the mental image of letter X, that produced a passage from isolated models to a general model in the meaning of Hejny (2004): children changed their cube construction way, they started from a face, putting cubes following the ‘X disposition’ (Fig. 7) and completing the remaining parts. Using the two diagonal’s theorem, the play becomes easier: the construction of a coloured village changes a lot, since starting from diagonals, the placement of the other houses is obliged.

Y			Y		B	Y	R	B
	Y			Y		B	Y	R
		Y	R		Y	R	B	Y

Figure 7

In other words, the finding of two diagonal’s theorem caused the passage from ‘the various way to make something to what it needs make’ in sense of Altieri Biagi & Speranza (1981) and also it produced the discovery of a common structure in the villages (Hejny, 2004).

Afterwards pupils found also that on the lateral surface of the cube there are three points (blue or red) that make a continuous and close paths. This was the input for another play, named ‘Cricket play’ (we prefer do not present it here), that conduced to find a ‘new’ theorem: “In the cube there is an “internal diagonal” with only one colour and the other diagonals of cube are of three different colours” (Fig. 8). In this way the analogy with the 2D play in the village emerged and the “small ladder’s theorem” reappears… Is it generalization?

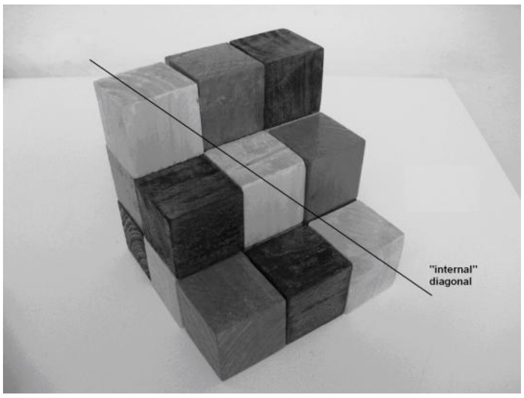

Figure 8: ‘Internal’ diagonal of cube.

CONCLUSIONS

We think that our experiment realized a good equilibrium between playing and learning, in particular we understood that play furnishes the opportunity to observe mathematical reasoning’s development in young pupils.

In reference to our first research question, we can reply affirmatively, concluding that in some kindergarten groups we observed the spontaneous appearance of the metacognitive instrument of generalization, motivated by play and also by context. So, that confirms our initial hypothesis about the early use of generalization. In literature we haven’t found similar researches and results with so young pupils.

In fact, in relation to the use of generalisation, we had better results in kindergarten than in primary school. We pose a possible explanation: in kindergarten the play was entirely conducted from the teacher with the presence of researcher as observer, whereas in primary school the work was conducted from both, teacher and researcher. In the first case, the observer had the possibility to “peek and catch” some observations made from children, while the teacher was involved in the action. That allowed to take advantage of these suggestions and to use them in the following activities and conversational managements. In primary school, may be that working with a researcher, an unfamiliar person, influenced negatively the performances of pupils. So, the answers to the second research question, according to Schuler (2011), could be: “Potentially suitable materials and games need a competent educator with regard to didactical and conversational aspects”. In other words, the role of the teacher and a conversational management appeared determinant.

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THE GENERALIZATION OF THE MEASUREMENT CONCEPT IN KINDERGARTENS THROUGH THE BARTER MARKET

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This paper shows a proposal research that tries to describe how five-year-old children can learn the measurement process. The focus is on how everyday life experience can help children build mathematical concepts, especially the process of measuring, and how children learn to use a special scientific language.

INTRODUCTION

Kindergarten in Italy has now become an integrated system in evolution, characterized by the fundamental right to education. Therefore, the final goal of kindergarten education is to promote the development of independence, skills and good citizenship in children. All this is reflected in daily experiences when a child recognizes and communicates an understanding of fundamental activities and manages transactions with others. Moreover, the child learns to appreciate other points of view and to recognize rights and duties (NCTM, 2000; NRC, 1989; INC, 2007). This research tries to find whether measurement-related concepts can be introduced in kindergartens by letting children prepare food and drinks whose ingredients need to be measured in several ways. Our objective is also to see if children can seize the underlying differences and similarities between the use of different measuring instruments and units of measurement. The school undertook to send school materials to a school in India. To obtain these materials, the children prepared, packaged and "traded" food products. The experience of preparing food and beverages for this project taught them the concepts of weight, volume and length (preparing pasta, juices, blended drinks, pastry cream and chocolate rolls). Finally, in assigning a value to these products in order to exchange them for the school material children learned about numbers in relation to pricing.

CULTURAL REFERENCES

In the field of experience, specifically "speech and words", the National Curriculum Guidelines indicate among its goals the development of specific skills including that of "communicating to others your own reasoning and thoughts through verbal language, used in an appropriate way in different activities" (INC, 2007). We have wondered what is the relationship between everyday language and scientific language at this particular stage of a child's cognitive development. Exploration, observation and comparison in scientific activities can be used to support the development of language among children and between children and adults. Therefore, the problem of mathematical communication “depends at least as much on what we see as on other types of less abstract speech”. The question then concerns the “effectiveness of communication” and its mediators: semiotics, artefacts and visual (Sfard, 2009). “Equally important to the acquisition of mathematical ideas is the neural system that governs body movements” (Lakoff, Nunez, 2005). Some research shows that body movements can express the perception of objects and spatial orientation and therefore crucial elements of mathematical reasoning. Dealing with the problem of measurement in kindergarten leads to particularly complex experiences and language. Moreover, Vygotskij’s development theory entrusts schools with the task of “stimulating” the movement from spontaneous to scientific concepts; on the one hand, this “stimulation” provides for the maximum development of the scientific concept acquisition stage, while on the other hand it exploits spontaneous concepts in order to promote the highest levels of cognitive development (Vygotskij, 1984). We can see, therefore, that “measurement can constitute an area of near development in which experiences, although not completely understood by a child, can successively be integrated into a network of conceptualization” (Bartolini Bussi, 2008). Moreover, it seems important, once again, to affirm that the learning objective in kindergarten is to enter the world of adults by following the “who, what, where, how, why” method in order to make a concept clear and to explain the meaning of a process (Ginsburg, Pappas, Seo, 2001). This objective can be realized by resorting to well-defined mathematical concepts, such as the ability to invent and plan, make similarities and relationships, as well as to analyse the different forms of natural language that are the starting point of every activity of formalization. It seems to us that we have followed the guidelines related to everyday activities, knowledge of personal history, time rhythms and cycles, space orientation and exploration of nature. It also seems to us very relevant to point out the importance of gathering, arranging, counting and measuring by resorting to more or less methodical ways of comparing and arranging, in relation to different properties, quantities and events through the invention and use of objects or sequences or symbols to record and remember some simple measuring instruments and, finally, by making quantification, numeration, comparisons (Geary, 1994; Ginsburg, Seo, 2004; Clements, 2004; Copple, 2004).

METHODOLOGY

The didactic methodology uses the inquiry approach, a model based on assumptions of knowledge, learning and teaching derived from criticisms of the traditional method of transmission. Through the inquiry approach, it is possible to: encourage students to explore; help students to verbalise their mathematical ideas; bring students to understand that many mathematical questions have more than one answer; make students aware that they are capable of learning mathematics; and, teach students, through experience, the importance of logical reasoning. In other words, we try to enable students to develop the mathematical capabilities necessary to pose and solve mathematical problems, to reason and communicate mathematical concepts and to appreciate the validity and the potential of mathematical applications (Borasi, Siegel, 1994). This has been recommended in numerous important American and Italian studies on reforming the teaching of mathematics (NCTM, 2000; INC, 2007).

Several researchers who have studied the learning of mathematics have found that students must actively demonstrate a personal understanding of mathematical concepts and techniques. Only in this way can they reach a level of significant understanding (Ginsburg, 1983; Steffe, von Glaserfeld et all, 1983; Baroody, Ginsburg, 1990). This position is reflected in constructivism. The influence of constructivism on mathematics teaching can be seen in requests for teaching environments that encourage students to actively participate in developing their knowledge rather than receiving it from teachers or books. In these classes, the roles are reversed. Instead of passively listening, the students assume responsibility for their learning. The teachers, on the other hand, speak considerably less and listen a great deal more to the students' reasoning in order to help them understand what they have deduced (Confrey, 1991). In other words, to be good students, children today must be researchers ("inquirers"). Therefore, only doubt and uncertainty can motivate the search for new knowledge (Skagestad, 1991). Our experience was based on the inquiry approach model, which allowed us to alternate problem posing with problem solving. It showed children solving problems which arise and for which no one has the answer rather than solving problems prepared by the teacher. For example, when they have to assign a price to one of their products, they decide on the basis of their different personal daily experiences. We have then chosen to get children to make some types of food such as pasta, cream, fruit juices and chocolate roll; in this way they can form their own opinion about the best way to measure things, not to mention the experience they have already gained from their everyday life.

This model led us to use the problem posing method in which the children's answers, their questions and the data they used are analysed. In other words, with this methodology the children can make observations, ask questions and formulate proposals. Moreover, they can compare an external investigation with an internal one. It is also possible to compare and contrast exact and approximate investigations, using the strategy of "and what if…" to generate new hypotheses. It has especially been important to see how children know special terms and the two main aspects connected with the measurement process: i.e. comparison and order. That's the reason why it was useful to analyse the clinical-like conversation not with a view to verifying the correctness of the answers but rather to gain an understanding of the social and cultural motivations behind them. It was an extremely important method for forming, informing and maintaining the teacher's "intermediary inventive mind" (James, 1958).

OUR RESEARCH, ITS RESULTS AND THEIR ANALYSIS

Our research has been carried out in two classes of two different kindergartens. In the first class there were 16 children and in the second 19; all in all, the project lasted 35 hours. One of the kindergartens was twinned with a kindergarten in India. The children saw films of this school and with the teachers decided to send school materials to the students there. From this came the idea to organize a "market" whereby the children traded the food and beverages they had prepared for pens, exercise books, etc. to send to India. The aim of the research was to give the student an enjoyable experience in which to experiment with measurement and then to relate it to their primary needs ("the right to food") and their childish pleasures. This situation turned out to have a great influence on scientific learning; in particular, it allowed children to become familiar with the concepts of weight, volume and length. This establishes a connection between children and the "who, what, where, how, why" method (Ginsburg, Pappas, Seo, 2001) and leads them towards the scientific conceptualization of the measurement process (Bartolini Bussi, 2008).

Through the presentation of some objects (a stick, an orange, a piece of chalk, a pencil, a coloured ribbon, some coins, a sheet, a bottle, a glass) we have tried to understand what children know of the size, weight and volume of these objects.

Children have then been spurred to have a clinical-like talk like the following:

> Teacher : Is the pencil longer than the chalk? Is the pot higher than the orange? Is the pot larger than the bottle? Is the orange heavier than the sheet?

After looking at the objects put on the desk children have started to express their opinion as follows:

> Mattia: The pencil is longer if I put it this way, while if I turn it the pencil is short!
>
> Federica: The bottle contains more milk than the glass!
>
> Giovanni: The orange is heavier than coins.
>
> Mattia: I'm taller than the stick, but Federica is shorter than me!
>
> Teacher: Which are the longest things you know? Which are the widest ones? Let's try to find the longest, widest, highest and thinnest things in this classroom.
>
> Mattia: The door is tall! ... and the teacher too, because she's taller than me!
>
> Giovanni: On the contrary, the window is wide.

The distribution of strips of paper having different length to each child has allowed us to make some inquiries about their previous intuitive knowledge of comparisons and orders. In particular we have asked children to find in the classroom some objects as long as their strip of paper.

> Mattia: My strip is as long as Luisa's case on the zip side.
>
> Federica: On the contrary, my strip is as long as the poster which leaves are stuck onto… it is very long!
>
> Teacher: This means that the poster which leaves are stuck onto is longer or shorter than Luisa's case?
>
> Giovanni: I think that Luisa's case is shorter than the poster which leaves are stuck onto because the strip of Mattia is shorter than the strip of Federica.

Then we have made accurate inquiries about the order concept by asking children to find the longest and the shortest strips so as to arrange them in length order, from the shortest to the longest. It's at this stage that we can infer how visual and artefact semiotic mediators become an important instrument for their "effectiveness of communication" (Sfard, 2009). After the talk stage the activity carried out at school concerning the above-mentioned objectives developed in three further stages: an initial observation and exploration stage of the actions and movements of an "expert" adult in the preparation of sweets; the second stage in which the children become cooks and, handling the ingredients, they formulate and verify hypotheses, because they have to reconstruct the previously observed procedures, going through the recipes and proving their validity; the last stage in which the attention is focused on the possibility to set up a trade fair as a problem solving exercise concerning the "value" of the prepared products and the meaning of fair exchange, identifying the objects to trade and their value. All the activities performed show how the inquiry approach is carried out in real terms and draws attention in particular to the formation of concepts according to the constructivism theory in the teaching of mathematics (Steffe, 2004).

In particular, in the first stage, three adult experts were brought in to prepare single products: a grandmother for the preparation of an ear-shaped pasta (orecchiette) typical of their region; a mother to make a cream pastry and a blended drink; and, a professional pastry chef to prepare a chocolate roll. After watching the experts prepare the products in class, based on typical housewife measurements such as "a handful of sth", "a pinch of sth" and "a spoonful of sth" there was a fruitful discussion on what they had observed. Problems relating to weight emerged when trying to interpret recipe indications given by a grandmother, such as "a handful", and the additional problem of the different quantities of flour contained in a child's hand and an adult's hand. Children of the two schools have solved the problem in one or more ways also thanks to the use of different instruments. A scale with two plates was used in one school; the following discussion ensued:

Teacher: "What is happening?"

Denise: The amount of flour in my hand is smaller and the plate stays up but Grandma's handful is heavier.

Teacher: Could we put the plates at the same height?

Giovanni: Let's put some other handfuls of flour on the plate to make it go up.

Teacher: Ok, but how much flour do we have to add?

Giovanni: As many handfuls as the two plates are at the same height [and he shows the height with his hands].

In the other school Mattia realizes that the amount of flour hold in each handful is different and says:

Mattia: … but the amount of flour is different, … I mean, it's more than my Grandma's handful, yes but my handful is smaller.

Mattia tries to convince his friends of the truthfulness of his statement and says:

Mattia: Let's take two sheets and let's put my Grandma's handful of flour on one sheet and my handful of flour on the other one. Look, it's more! Look!

Federica: Yes, it's true, you're right!

It is possible to infer from what children have said two main aspects of the measurement concept at intuitive level, i.e. comparison and the additive principle between homogeneous quantities. The inquiry method is also reflected in this conversation, as there are a lot of solutions to the same problem and also the desire to support their opinions.

The importance of the linguistic aspects in the relationship between natural language communication and mathematical communication became as clearly evident as did the problem of learning mathematical concepts through body movements (Sfard, 2008; Lakatoff- Nunez, 2005).

When preparing the blended drinks and the pastry cream, the "expert" indicated the necessary quantities of ingredients but the children had to choose the proper instruments to measure the liquids and solids. For example: a big glass indicated a greater quantity of milk than a small glass which the children discovered contained exactly half the amount; a soup spoon rather than a teaspoon was used to put more sugar in a drink; a ladle contained even more than a soup spoon. The practical experience of preparing pastry cream and blended drinks involved the children in a discussion of volume-related units of measurement. With the expert, the children decided which utensils (soup spoons, teaspoons, ladles, big glasses, small glasses) should be used to measure the ingredients.

Mum and cook: Right, let's see children which utensil is better according to you? Take a look at these utensils (the mum shows the spoon, the teaspoon, the ladle, the glass, the cup, and so on).

Vincenzo: Let's measure the flour with a ladle because it holds more than a spoon which can be used to measure sugar.

This started a discussion on the quantity of liquid already prepared which, according to Federica, would not be sufficient for everybody once it became cream.

Federica says: But we haven't got enough cream for everybody!

Mum: But why? How can you say it?

When she was asked how she could be sure of this, she suggested dividing the cream among all the students. Upon verifying that there was only enough cream for half the students, she suggested adding double the amount of ingredients to the mixture. When they finished preparing the cream, they started looking for ladles to pour the cream into glasses and decided to pour four ladles into the big glass and two into the small glass. The children were able to see the change in volume between a glass of a substance before being blended and after. During the preparation of the blended drinks the children first invented and produced the recipes discovering the changes in volume between the quantity in a glass before and after it was blended. They filled a big glass with pieces of fruit, milk, orange juice and sugar but once it was blended the volume increased producing enough liquid to fill a big glass and a little glass. Another interesting aspect emerged during the preparation of the chocolate roll. This product was chosen to study a series of questions related to the concept of length which was dealt with in a natural way by the children during the activity. The ensuing discussion allowed the children to come to a common understanding. Then, the natural desire to eat the chocolate roll led to find a way of dividing the roll in equal parts. The teachers had equipped the classroom with "good" instruments for measuring and the children, looking around the classroom for something to help them measure, were able to identify instruments long enough for this purpose. Next, the children chose a strip of paper as the best tool for measuring and then they developed a way of folding the paper in equal parts. This folded strip was then used to cut the chocolate roll into enough equal parts for all the children. The direct experience of preparing the chocolate roll was planned as a problem-solving activity concerning the concept of length. The children managed to devise and execute a system for dividing the roll in equal parts for everyone thereby learning the concept of multiples. Mattia and Pietro try to compare the lengths of different sheets of paper scattered on the table to the length of the chocolate roll. When he finds one that was just slightly longer than the roll, Federica says, "Let's cut off the extra bit and write "The Length of the Chocolate Roll" so that we know which is the right piece!" Mattia suggests using the strip as if it were a ruler by putting measuring marks on it but the idea proves to be difficult to apply because the marks do not allow for cutting equal pieces. Mattia has another idea. He suggests folding the strip in two but the chocolate roll is still longer and bigger and, if cut in this way, there would only be enough for two children. In fact, Mattia measures the folded paper against the chocolate roll to see if it is exactly half the length and verifies that it is. Then Federica suggests, "Let's fold the strip in half again" but it still is not enough for everyone. The children continue folding the paper strip until there are enough pieces for everyone. This discovery gave rise to interesting games on the meaning of double and half using other materials.

This way we have made inquiries about the possibility that children can develop the ability to face situations of problem solving and problem posing. Moreover, it is obvious that children have been able to take an indirect measure using the instrument of the semiotic mediation (i.e. the strip of paper) and so they have found a way of making an effective unconventional "metre" (base 2) having submultiples, too (principle of Eudosso – Archimedes).

In the second stage, when they personally prepared the baked goods and had to deal with measurements, the children had the opportunity to experiment with the concepts of weight, length, and volume. During the preparation stage we have observed how children can get the main concepts of the measurement process, even if at an intuitive level. As for the preparation of orecchiette, for example, children have to make a measurement roughly and at the same time more and more accurate, which doesn't mean that this measurement does not follow a definite plan or pattern. Preparing the cream leads children to make comparisons thus choosing the suitable measuring instruments; and as for the preparation of the chocolate roll it is necessary to use adequate units of measurement.

In the third and final stage, the children organized and operated a barter market where they "exchanged" their goods for school materials on the basis of "price lists" which they had developed and agreed on previously. The expression "Barter-Exchange" was introduced at the beginning of the project, during the preparation of the orecchiette. The teachers bring to the children's attention that the grandmother had worked hard to make the orecchiette and should be compensated for this. Our goal was to get the children to barter in exchange for school materials to send to the Indian school twinned with ours. This experience led to the organization of the barter market and the development of the "price" list. In establishing the "prices", it is important to emphasize the process by which the children attributed value to their products. For example, "*if a complete chocolate roll was worth a package of ten exercise books, then how much was one piece of chocolate roll worth?"*. In choosing which products to exchange, it was necessary to use the concept of double. For example, the children agreed among themselves that a small glass of pastry cream was equal in value to exactly half that of a big glass and a big plate of orecchiette was worth double the amount of a small plate. To determine the value of the blended drinks, the children took into account the preparation time and the change in volume and therefore the need to ask for more school materials in exchange.

In order to exhibit the price list to the public, some kind of poster was necessary. The children solved the problem by designing one with drawings of all the instruments used to measure the various products: glasses, espresso cups, different kinds of plates and the short and long strips of paper. During the fair, each child bartered their products with the adults and, at the same time, explained how the products were prepared and, above all, how they arrived at assigning a value ("price") to the products. In this way, it was possible to verify that the child had acquired a full understanding of both the concepts related to measurement and the value attributed to the products. The observations relative to the price of the products are equally interesting. Initially, the children were reluctant to barter because of their personal feelings for the objects they had made, a behaviour that is typical in this age group. This strong personal attachment to the products was further highlighted in the barter stage when there was a request for a "big" piece of cake, for example. It was observed that often the value of an object was closely tied to its size. During the barter market another concept linked to the measurement process was examined. When children had fixed the price of each product the equivalence between different units of measurement, as well as the main concepts of the equivalence, have come out once again in an intuitive way. The drawings made by the children in the price list are evidence of how children have acquired the above-mentioned concepts of measurement. The entire project proved to have embraced all the fields of experience included in the Italian curriculum guidelines, not only the specific one related to mathematics and "knowledge of the world".

CONCLUSION

This experience allowed us to confirm the idea that it is possible to talk about a child's scientific knowledge, as long as we give this sentence the right connotation. To avoid making an "intellectual mistake", we must talk about "correct knowledge". This is what our research in kindergarten is generally devoted to: having the child's first experiences and reasoning follow a "correct" formulation, always respecting the development of the child, who must not be thought of as an "adult". Moreover, what we continue to observe in our research, and what stimulates and supports us, is the children's sincerity when facing different situations, that spiritual condition which prevents them from wanting to distort the observed reality, their capacity to ask questions without feeling judged, as well as their ability to change their mind. These are all typical of children's behaviour (something which adults no longer have) but they are also essential requirements when talking about "scientific nature".

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