Supporting Independent Thinking Through Mathematical Education

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Introduction

Mathematical education is recognized internationally as central to society. The teaching of mathematics begins at a young age, because basic mathematical concepts are at the heart of both personal and social development. There is no doubt that issues connected with mathematics education need to be at the center of attention of political leaders, educationalists the general community, and, of course, parents and teachers.

Teaching mathematics is important both for supporting the development of the child and for solving critical problems in a global society. Number sense, numerical literacy, spatial abilities and other fundamental skills and concepts of mathematics, are critical to social and personal growth and understanding. By means of it, mathematical knowledge gained with the help of teachers can favour the pupils with logical thinking and reasoning, which aids the conduct of dialogue and negotiation. In this way, mathematics supports ethical behavior, especially understanding human rights and obligations. The ability to organize and use data is valuable in almost all spheres of individual and social life. The search for solutions promotes creativity, flexibility, and adaptation to new situations, and succes in finding (multiple) solutions supports the development of self-esteem.

The quality of teaching and learning mathematics depends on many elements, affected and determined by each other. While many factors, such as social structures of inequity and diversity, are seemingly beyond the remit of the individual teacher, he or she remains a central element, responsible for what is going on during lessons in their classroom. Teachers must understand their role, both within the classroom, and as a part of larger social and political structures. They must blend their interactions with pupils and their understanding of mathematical content objectives with their own ethical and moral commitments in order to effect change in society.

Teacher-training in mathematics goes far beyond subject-specific and pedagogical content. It connects with many other realms: psychology (creation of concepts, emotions, motivations, interactions, ...), linguistics (communications, language in learning and teaching mathematics, symbol creation and its understanding, ...), socio-cultural theory (ethno-mathematics, equity and diversity, ...), history and epistemology (developments of mathematical concepts, historical obstacles in understanding mathematical concepts, ...), technology (application of technology in mathematics, using computers in teaching mathematics, ...), and so on.

Few people enter the field of teaching with a comprehension of the complexity that such work entails. The education of the teachers of the future, and the ongoing professional development of practicing teachers must help them to negotiate these complexities and to reconcile the potential conflicts between the realities of teaching and their own personal moral and professional commitments.

Open questions
in mathematical education

Part 1

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Sense and representation in elementary mathematics

Two concepts are central to contemporary mathematics education theory and practice: the support of sense-making by pupils, and support for developing facility with representations. This presentation problematizes and recasts both of these concepts by framing learners as artists – creators and producers – within a curriculum that usually wants them to "consume and use" instead. A common assumption is that mathematics curriculum is content that represents and interprets. Applying work of Sontag that argued against representational art, we can generate new forms of learning activities where artists evoke parody, abstraction, decoration, and non-art in ways that make mathematics vibrant and relevant to several of our conference themes.

The Introduction of our book calls our attention to the ways that our efforts are tools for both the development of the child and for solving critical problems in a global society. Indeed, these two ways of thinking about our work are always interwoven, since individual children are always present and at the same time future members of our society. That is, our global society is nothing other than ourselves. Education in general, and mathematics education in particular, is central to the basic existence and aims of social life. I start today by reminding us that this introduction is, if anything, gentle in its call for principals, education officials, the general community, and, most importantly, parents and teachers, to consider how and why fundamental mathematical concepts are at the heart of both personal and social development. Toward this aim, I ask us to think about two concepts that undergird most of the intellectual work of teachers and curriculum workers in mathematics education, and to think critically about the ways in which they impact on our beliefs about what we should do and what might be changeable: the support of sense-making by pupils, and the overarching aim of facility with representations.

By 'sense-making', I am referring to the common assumption that our task as mathematics educators is to help young people to make sense of mathematics. We receive mathematics as a reasonable and logical world within which one should be made to feel comfortable and secure. We often agree that mathematics is the one place where we can be certain about what we know and whether or not we are correct. My comments today have to do with the power of these assumptions to enable specific kinds of educational experiences while also perhaps failing to allow pupils to fully appreciate the wonders and powers of mathematical modes of inquiry and understanding.

With 'representations', I ask us to consider also the power of our typical pedagogies, which tend to lead students from the concrete to the abstract, and also to move students away from specific instances of mathematics in the world toward general representations of these instances. We might think of the representations (of the ideas) as the actual material and content of the mathematics itself. I mean here simple things like numerals to represent numbers of things; drawings of shapes to represent ideal geometric relationships; fractions to represent parts of wholes, proportions, and ratios; equations to represent functional relationships, letters to represent variables that may take on different values, and so on. Other representations model mathematical concepts and relationships, such as base-ten blocks for arithmetic operations, drawings of rectangles or circles for fractions and ratios, or graphs which visually represent algebraic equations. In my experience, much of mathematics education aims to help students to develop artistic virtuosity with mathematical representations for communicating their ideas. However, if we take this artistic virtuosity seriously, then critics of artistic practice sometimes suggest that representation is not always the aim of art, and in fact, representation often violates art itself. What could young children, as mathematical artists, do, then, if they would not primarily be practicing forms of mathematical representation? I will return to this question, because it is connected most directly with our Introduction's call to consider the broad, social contexts even as we focus on the individual mathematicians in our kindergartens, primary school classrooms, and on the adolescent mathematicians with whom we work day-to-day.

Stop making sense

Brent Davis (2008) recently wrote, "In the desire to pull learners along a smooth path of concept development, we've planed off the bumpy parts that were once the precise locations of meaning and elaboration." We have, he says, "created obstacles in the effort to avoid them." Davis describes "huh" moments, when it is possible to enter authentic mathematical conversations. For example, we might ask someone to describe what we mean when we write '2/3 = 14/21'. Responses vary from pictures of objects to vectors on a number-line, but all share a conceptual quality of relative change so that increasing one thing leads to a proportional increase in another thing or group of things. However, when we ask the same person to describe what is happening in the expression '-1/1 = 1/-1', we usually get a kind of "huh", which communicates a moment where the mathematics has lost its sense, but which also potentially begins an important (mathematical) conversation. In my own work on what Davis calls the "huh" moments – when mathematics stops making sense to us, and we grope for models apparently not available (Appelbaum, 2008) – I, too, have noted the potential for the non-sense-making characteristics of mathematics to generate different kinds of teacher-student relationships, and most significantly, different kinds of relations with mathematics within associated critical mathematical action (Appelbaum, 2003). Mathematics curriculum materials too often hide the messiness of mathematics where sense dissolves into paradox and perplexity, but more importantly they construct a false fantasy of coherence and consistency. As most professional mathematicians understand, mathematics at its core is grounded in indefinable terms (set? point?), inconsistencies (Gödel's proof? Cantor's continuum hypothesis?) and incoherence (the limit paradox in calculus?). At a more basic level, multiplying fractions ends up making things smaller even though 'multiplying' conjures images of 'increasing' to many people; two cylinders made out of the same piece of paper (one rolled length-wise, on width-wise) have the same surface area but hold different volumes; we're taught to add multiple columns of numbers from right to left with re-grouping, when it is so much easier to think left to right starting with the bigger numbers. In some cases, it is impossible, speaking epistemologically, for mathematics as a discipline to 'make sense'; in others, it might be more valuable pedagogically to treat mathematics as if it does not make sense. To do so would celebrate the position of the pupil, for whom much of the mathematics is new and possibly confusing anyway.

Yet, so much of contemporary mathematics education practice is devoted to helping students make sense of mathematics! What if, instead, we stopped trying to make sense, and instead worked together with students to study the ways in which mathematics does and does not make sense? Instead of school experiences full of memorization and drill on techniques, we would imagine classroom scenarios full of conversation about the implications of one interpretation over another, or of explorations that compare and contrast models and metaphors for the wisdom they provide.

Elizabeth de Freitas (2008) describes our desire to make mathematics fit a false sense of certainty as ‘mathematical agency interfering with an abstract realm’. She encourages teachers to intentionally ‘trouble’ the authority of the discipline, in order to belie the ‘reasonableness’ of mathematics. In this way, we and our pupils can better understand how mathematics is sometimes used in social contexts like policy documents and arguments, business transactions, and philosophical debates, to obscure reason rather than to support it. Stephen Brown called this kind of pedagogy, “balance[ing] a commitment to truth as expressed within a body of knowledge or emerging knowledge, with an attitude of concern for how that knowledge sheds light in an idiosyncratic way on the emergence of a self” (Brown, 1973, p. 214)

So, you may wonder, what does this mean about curriculum materials and textbooks? “Obviously somebody somewhere with a lot of authority has actually sat down and written this Numeracy Strategy,” says one teacher with whom Tony Brown (2008) spoke. “it’s not like they don’t know what they are talking about.” Tony Brown blames the administrative performances that have shaped mathematics for masking what Brent Davis calls the huh moments, and what de Freitas describes as the self-denial that accompanies “rule and rhythm”. Teaching in this “senseless world of mathematical practice” need not abandon science and the rational. It merely shifts teaching away from method and technique toward what Nathalie Sinclair calls the “craft” of the practitioner, as she evokes the metaphor of teaching as midwifery from Plato’s *Theaetetus* (see also Appelbaum 2000). As midwives, teachers assist in the birth of knowledge; students experience not only the pain and unpredictability of the creative process, but also the responsibility for the life of this knowledge once it leaves ‘the womb’. One must care for and nurture one’s knowledge, whether it acts rationally or not. Can we be confident that the ways we have raised our knowledge will prepare it for when it is let loose upon the world? Will our knowledge be embodied with its own self-awareness and ethical stance?

A dubious theory

A demand that everything make sense, and that this sense be so simple that it is virtually instantaneous if at all possible, dominates the way we work with mathematics in school. We design a curriculum that introduces a tiny bit of new thought once per week or even less often, because we worry that a pupil will feel lost or confused, and not be able to move on to the next tiny new step that follows. I imagine instead a curriculum where children beg for new challenges, and where these children delight in the confusion that promises new worlds of thinking and acting, of children we do not just ‘get by’ in mathematics class, but who love mathematics as part of their sense of self and their engagement with their world. The French philosopher and social theorist Michel de Certeau (1984) blamed the social sciences for reducing people to passive receivers of knowledge. And indeed, educational research and practice has been dominated by the social sciences for the past century, so we have been living the successes and failures of these approaches to education and now need to look at them critically as we reassess our work in mathematics. de Certeau suggested that the social sciences cannot conceive of people as actors who invent new worlds and new forms of meaning, because they study the traditions, language, symbols, art and articles of exchange that make up a culture, but lack a formal means by which to examine the ways in which people re-appropriate them in everyday situations. This is a dangerous omission, he maintained, because it is in the activity of re-use that we would be able to understand the abundance of opportunities for ordinary people to subvert the rituals and representations that institutions seek to impose upon them. With no clear understanding of such activity, the social sciences are bound to create little more than a picture of people who are non-artists (meaning non-creators and non-producers), passive and heavily subjected to ‘receiving’ culture. Social sciences thus typically understand people as passive receivers or “consumers” rather than as makers or inventors of culture, ideas, and social possibilities. Indeed, I believe this is exactly the situation we find ourselves in as we seek ways to make mathematics meaningful for young people and for young people to take advantage of mathematical skills and ideas as they participate in their local and global communities.

This kind of misinterpretation is critical to our „consumer culture,” in which people are assigned to market niches and sold products, concepts, modes of life, and predictable desires. In curriculum as in advertising, such social science persists, so that we see students as consumers of knowledge whose desires are shaped by the curriculum via the teacher, teachers as consumers of pedagogical training programs, and so on. de Certeau employs the word „user” for consumers; he expands the concept of „consumption” to encompass “procedures of consumption” and then builds on this notion to invent his idea of „tactics of consumption”. School curriculum tries to sell students on the value of mathematical knowledge; we sometimes call this ‘motivation’. New curriculum materials are published and sold as part of a global economic system that demands new and improved products in a cycle of perpetual obsolescence and innovation.

What would it mean for youth who are learning “stuff that many adults already know” to be artists – creators and producers – when we seem to want them to “consumer and use” instead? The critical notion turns out to be how we make sense of the “art.” Susan Sontag (1966) wrote about what she named a “dubious theory” that art contains content, an approach that she claimed violates art itself. When we take art as containing content, we are led to assume that art represents and interprets stuff, and that these acts of representation and interpretation are the essence of art itself. Likewise in school curriculum, we often imagine the curriculum as content, and move quickly to the assumption that this curriculum represents and interprets. This makes art and curriculum into articles of use, for arrangement into a mental scheme of categories. What else could art or curriculum do? Well, Sontag suggests several things: To avoid interpretation, art may become parody. Or it may become abstract. Or it may become (‘merely’) decorative. Or it may become non-art.” (Sontag 1966: 10)

New worlds of mathematics education

Parody, abstraction, decoration, and/or non-art are three types of tactics for art and curriculum. I think, too, that they can be used to stop making sense of mathematics *for* young children, and instead, in the words of our introduction, they can help us ‘not only to pose questions, but also to look for solutions’. Common work of our book is focused around four main issues: Mathematics as a school subject; Teacher-training; Teachers’ work; and Learning Mathematics. I conclude with a brief outline for applying the de-Certeauian-Sontagian ‘tactics’ in each of these four realms. With my suggestions, I am encouraging each of us to consider how school mathematics could be experienced as something *other than* a representation of content, or something other than an abstract representation of ideas. This does not mean that I want us to abandon representations or the representation of ideas, but that our methods of teaching would not stress this as our primary purpose.

Mathematics as a school subject: Normally, we emphasize two kinds of experiences in school mathematics, and through these we create an implicit story about what mathematics ‘is’. We either develop ideas out of concrete experiences, or we model real-life events with mathematical language. An example of the first would be to work with numerals to represent numbers of objects, in order to stress for young children the differences between cardinality and ordinality, or to develop arithmetical algorithms for adding, subtracting, multiplying or dividing numbers.

We might work with base-ten blocks, number lines, collections of objects, drawings of objects, and so on. An example of the second might be to create a story problem out of a real-life situation, such as to ask how many tables we need for a party if each table can seat six people, and we expect fifteen people to attend our party; or, to ask, given eighty meters of fencing material, what shape we should use to have the most area for our enclosed playground. Now, suppose we wanted to transform our pedagogy so that the work in our classroom were one of parody, abstraction, decoration, or non-art, rather than representational art. Children might parody routine questions by acting out seemingly absurd situations where the reckoning leads to ludicrous results, or they might ask and answer questions that shed humorous or critical light on typical uses of the mathematics. For example, 4-year-olds who have counted the number of steps from their classroom to the door of the building, in ones, threes, and fives, might then count the number of drops of water to fill a bucket in ones, threes and fives, even though it seems to make no sense to do so … this would only be a parody, though, if the children themselves suggested it as a silly thing to do that they wish to do nevertheless. Similarly, ten-year-olds might design alternative arrangements of their classroom that make use of unusual shaped desks, such as asymmetric trapezoids, circles, etc. Mathematics might be abstract if children did more comparing and contrasting of questions, methods, and types of mathematical situations, rather than focusing on the particular questions or on practicing specific methods. For example, 8-year-olds might first organize a collections of mathematics problems first into three categories, and then the same problems into four new categories, rather than solving the problems themselves; the classification of the problems into types would constitute the mathematical work, rather than the solution of the mathematical problems. Mathematics as 'decoration' might be accomplished through a classroom project where students experiment with different representations of a mathematical idea for communicating with various audiences. After working with ratio and proportion, for example, a class of 11-year-olds might form small groups, one of which creates a puppet show for younger children, one of which composes a book of poems for older children, and another of which prepares a presentation for adults at their neighborhood senior citizens community center, all on the same subject of applying ratio and proportion to understand the ways that a recent election unfolded. In this sense of considering the appropriate way to describe ratio and proportion for a particular audience, the mathematics is more of a decorative from of rhetoric than a collection of skills or concepts; the important concepts have more to do with democratic participation in elections than with the mathematics per se. Mathematics as non-art uses artistic work that is not considered 'art' as its model – we could ask, when is creative mathematical work not mathematics? One answer is, when it is something else other than mathematics per se – for example when it is an argument for social action presented at a meeting; when it is an example used to demonstrate a philosophical point; when it is a recreational past-time; etc. In other words, mathematics as non-art would be mathematics not done for its own sake; mathematics as non-art would be mathematics for the purposes of philosophy, anthropology, literature, poetry, archaeology, history, science, religion, and so on. As long as the activity has purpose other than the mathematics itself.

Teacher's Work: So, mathematics as a school subject can and should take on the character of parody, abstraction, decoration or non-art. If this is to occur, there are important implications for the teacher's work. For one thing, the teacher would not be providing clear presentation or explanations of mathematical concepts or procedural skill. Instead, we can learn from current work at the University of Amsterdam on the types of teacher help that support mathematical level-raising (Dekker & Elshout-Mohr 2004, 2005; Pijls 2007; Pijls, Dekker & Van Hout-Wolters 2007). In their studies, they have found that teacher help directed at mathematical content –

explanations and demonstrations, is rarely more valuable than teacher help directed at collaborative learning and groups processes, and in fact sometimes teacher help focused only on the group processes leads to more significant conceptual level-raising. In other words, the nature of useful teacher work involves making it possible for pupils to participate as creators and consumers of mathematical art that is not representational, and which does not aim at simplifying the path to sense-making. Instead, teacher work essentially makes it possible for pupils to experience together the authentic practices of sense and non-sense through events such as parody, abstraction, decoration, and non-art. In the group processes that are supported by teacher-help, the mathematics is secondary to the group process in the teacher's mind. The teacher is helping the pupils to use mathematics in order to accomplish the group process, rather than using organization of the group in order to accomplish representation or sense-making of mathematics. This seems backward, given that our job is to teacher mathematics! It is almost counterintuitive! But, indeed, when we think this way, perhaps, there is a new "sense" to be made of mathematics teaching.

Teacher-training: What, then, are the implications for teacher training? I believe the key things to think about are the differences between preparation for representation and sense-making, which has been the primary direction of mathematics education for the last century, and preparation for the support of artistic practice. We have inherited a technology of teaching methods steeped in cognitive psychology which direct the teacher's attention to individual cognitive development. This has certainly been useful, and will continue to be useful to all of us in our work. However, I am suggesting today that we foreground another orientation to our work, which Eliot Eisner (1991) called criticism and connoisseurship. Ordinarily, the teacher training that I am most familiar with involves extensive practice in the application of methods, diagnosis and remediation. Eisner's ideas suggest instead that teachers-in-training spend more time immersed in experiences that are not directly focused on the representation of teaching and learning, or on making sense of what pupils can and cannot do, but instead on criticism and connoisseurship in the context of schools.

Connoisseurship is the art of appreciation. It can be displayed in any realm in which the character, import, or value of objects, situations, and performances are distributed and variable, including educational practice. The word connoisseurship comes from the Latin *cognoscere*, to know. It involves the ability to see, not merely to look. To do this we have to develop the ability to name and appreciate the different dimensions of situations and experiences, and the way they relate one to another. We have to be able to draw upon, and make use of, a wide array of information. We also have to be able to place our experiences and understandings in a wider context, and connect them with our values and commitments. Connoisseurship is something that needs to be worked at – but it is not a technical exercise. The bringing together of the different elements into a whole involves artistry.

It may sound like I am advocating an elitist notion here, but I do not mean this; indeed, I want us to think mainly about the depth of knowledge that all people have in their everyday lives as connoisseurs of those things they taste deeply, and to imagine how we could help young people to take those ways of learning and thinking and making meaning, and see that they are relevant in school (Gustavson & Appelbaum 2005; Appelbaum 2007). Now, what Eisner makes clear in his writing, is that educators need to be *more than* connoisseurs. They need to become critics. Our models for ourselves need to be those reviewers of films, albums, music videos, and video-games that we read and listen to for pleasure, and that help us to know which artistic works we will enjoy and find valuable, even those critics with whom we love to disagree. Criticism is the art of disclosure, of revealing more than the obvious; as John Dewey pointed out in his book *Art as Experience*, criticism has as its aim the re-education of perception. The task of the critic is to help us to see.

> Thus … connoisseurship provides criticism with its subject matter. Connoisseurship is private, but criticism is public. Connoisseurs simply need to appreciate what they encounter. Critics, however, must render these qualities vivid by the artful use of critical disclosure. (Eisner 1985: 92–93)

I see direct connections with our introduction, which describes teachers as crucial to the evolution of mathematics education:

> The quality of teaching and learning mathematics depends on many elements, affected and determined by each other. While many factors, such as social structures of inequity and diversity, are seemingly beyond the purview of the individual teacher, the teacher in the classroom remains a central element, responsible for what is going on during lessons in the immediate context. Teachers must understand their role, both within the classroom, and as a part of larger social and political structures. They must blend their interactions with pupils and their understanding of mathematical content objectives with their own ethical and moral commitments as a change agent in society.

So, in my own work in teacher education, I strive to work as a connoisseur and critic, in order to support the artistry of my students who wish to be teachers. And I welcome conversations with you over coffee, tea, a beer, wine, and so on, to share such stories. But back to the main theme of this presentation: what sort of mathematics learning is enabled by a teacher with extensive background in connoisseurship and criticism?

Learning Mathematics: Well, we could simply say, pupils of mathematics would be succeeding when they are demonstrating abilities to use mathematics in order to achieve a parody, to communicate an abstraction, as a decorative element in other contexts, or as non-mathematics across the curriculum. But more directly, I offer the following: Young people learning mathematics are artists whose tactics of parody, abstraction, decoration and non-art are forms of consumption that re-appropriate school mathematics as a tool of connoisseurship, and thus, of remaking their world anew in each act of mathematics they commit. Here is a very active and vibrant way to imagine mathematics learning: as artistry, as doing, as alive, and as transforming the world in every tiny moment. Mathematics in this "sense" is a collection of tactics for doing this. And learning mathematics is an apprenticeship in the artistry of social participation. Their mathematical actions, as art, are not aimed at a purpose that involves curricular illustration, but instead become the embodiment of critical pedagogy that engages both the mathematical artist and the artistic mathematician in critical citizenship (Springgay and Freedman 2007). I end, then, with a challenge to you: are you ready to allow the children in your life and work to become connoisseurs of mathematics? That is, to become more than knowers, to become critics of mathematics? Mathematics as criticism is an art of disclosure, of revealing more than what is obvious on the surface. Here is the magic recipe for achieving this: think more about coordinating activities where the children are active artists of mathematics than about how to represent or explain clearly a mathematical concept. I know, it goes against so much of our desires to make things easier for the child. In the end, though, if we stick to this plan, we will be lucky enough to spend time with current and future crafters of beautiful worlds, young people who use mathematics to shed insight on contemporary society, to ironically critique common sense practices, as tools for appreciating and interpreting culture and societal problems, as the medium of decoration and entertainment, and simply as so valuable as to be part of all things not usually named 'mathematics'.

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Factors hampering independent mathematical thinking

Examples of adverse educational effects of the false stereotypical public image of mathematics are discussed. While the role of the context of a task given to children in primary education is indispensable, it is argued that the role of understanding the purpose of the task by the children is also essential. Some ways of developing independent mathematical thinking through critical thinking and of stimulating the development without overdemanding are outlined.

Introduction

The subject of the conference is not only the question of supporting independent *mathematical* thinking, which by itself is very important, but it also provides a much wider perspective: mathematics education as a means to support *independent thinking in general*.

I will concentrate on obstacles which hamper the implementation of the idea of independent mathematical thinking and on some ways of overcoming them.

False stereotypical public image of mathematics and its consequences

Some thirty years ago I read a feature article on the front page of a leading Polish weekly *Polityka*, written by a well-known author. The title "I defend the students' right to do their own thinking" would fit the topic of our conference. However, I was astonished by his introductory remark: "Of course, this applies only to the humanities; in mathematics the student can only copy the teacher".

My immediate reaction was that the opposite statement would be more fitting: *at the beginning of schooling, mathematics is the only subject in which children can check basic facts on their own*. Generally, students have to accept most of school knowledge on the grounds that the teacher has said so or that "the book says so". This includes, say, the fact that the river flowing through Cracow is the same river as the one that flows through Warsaw; few students are able to verify this fact. Similarly, the information that the ruler of Poland accepted Christianity in 966 must come from external sources.

In contrast, the child, gently encouraged by the teacher, can use fingers to show that 5 + 3 is 8 and 7 + 5 is 12. Knowledge of such number facts does not have to depend on information given by the teacher or stated in a textbook. Similarly, children can find out that, say, 4 times 7 is 28 by repeated addition. Basing themselves on such experience and encouraged by the teacher, they can gradually memorize the multiplication table with the feeling that they have validated each of its items and can check them again if needed. Admittedly, when the subject matter becomes more advanced, such verification is only partially possible. Nevertheless, children's conviction that they can verify mathematical facts seems to be a crucial factor influencing their attitude towards the subject.

Misunderstanding of what is mathematics, illustrated by the above quotation from an essayist, is not uncommon. It is part of thinking of mathematics in terms of "musts", "dos" and "don'ts". Such an image (particularly conspicuous in case of school algebra) is often part of the hidden curriculum.

Teaching methods often do not distinguish between *laws* (such as, say, that of commutativity of multiplication) and *human conventions* concerning the system of notation. The fact that 4 times 7 equals 7 times 4 is true regardless of the symbols used to express it and will be true forever; the rule that in an expression such as $5 + 7 \cdot 4$ multiplication has priority over addition is not an objective law but a historically developed way of reading and interpreting such symbols, similar to the convention that the cross + denotes addition while the double bar = denotes equality.

Of course I do not mean that such subtleties (objective law versus notational convention) should be explained to children, but we should be aware of the potential obstacles involved. Let me illustrate the problem with some unexpected difficulties arising in apparently clear situations.

My first example concerns an excellent elementary teacher who felt (long ago) uneasy when she read a rule in a textbook: "If there is only addition and subtraction of numbers and there are no parentheses, the operations are performed from left to right" and compared it with the change of the order of operations: $24 + 37 + 6 = 24 + 6 + 37 = ...$, which appeared later in the same textbook (intended to show how commutativity may help someone with mental computations). She found the change of order incompatible with the previous rule. I then realized how serious was the problem. The source of her difficulty was the conception of the arithmetic symbol system as based on "must" and "must not" rules. The formulation "you *must* perform operations from left to right" is not what we mean. However, the formulation, say, "you *can* perform operations from left to right" is ambiguous. A theoretic explanation based on the concept of the value of an arithmetic expression would be too sophisticated for primary school students.

Second example. A preservice elementary teacher was to get credits for a math course. She got a series of arithmetic tasks together with the "responses" of a hypothetical child; she was to find out which computations were correct and to pinpoint errors, if any. One of the tasks was to judge the computation:

$8 \cdot 6 = (5 + 3) \cdot 6 = 5 \cdot 6 + 3 \cdot 6 = 30 + 18 = 48.$

The examinee has written that passage from $(5 + 3) \cdot 6$ to $5 \cdot 6 + 3 \cdot 6$ is incorrect, because the rule states that one first performs the operation in parentheses, that is, one must add 5 + 3.

Both examples vividly show what is lost in the "must-mustn't" image of mathematics. In either case the evoked rule ignores the *purpose* of a given step of computation (analogously, rules of the traffic code concern forbidden actions and not the destination of the journey). Such a misleading image of mathematics, resulting from years of inadequate rule-based school instruction, is very hard to change.

In Poland, in the 1970's, primary education was influenced by Zofia Cydzik, a leading educator and author of textbooks. She distinguished between a *way* and a *method* of performing an operation on numbers. A way depends on specific numbers and may be modified whereas a method is general and has to apply to all numbers. She argued that one should teach methods, not ways, because mastering a method guarantees success for all computations of a given type; a "way" may be easier for certain numbers but may fail in new cases. She also believed that teaching general methods fosters a true image of mathematics.

I mention this because her point of view, although erroneous, is not unusual and still affects education adversely. The scheme of presenting methods in Cydzik's textbook was to give a model example and then a series of tasks in the common format: "Imitate this example". One of the series devised for "going over ten" started with a *didactic template*

(*) $8 + 5 = 8 + 2 + 3 = 10 + 3 = 13$,

which was followed by analogous additions. The child was to copy the pattern using different numbers. The last item was $9 + 10 = 9 + 1 + 9 = 10 + 9 = 19$. Many parents-mathematicians found this irritating as a spectacular example of a nonsensical manner of adding 9 + 10.

Such teaching schemes resulted from, and contributed to a false stereotypical image of mathematics. In turn, this distorted view is one of main obstacles to students' independent mathematical thinking.

Didactic templates, albeit in more reasonable and less conspicuous forms, can be traced in many textbooks for primary and secondary schools; usually the ways of pushing arithmetic (or algebra, geometry, calculus, ...) are more subtle. Templates are convenient for teachers as workable schemes of what students should learn; their main disadvantage is that they fail to foster deeper understanding of the subject.

Before the 1960»s many educators insisted on showing children only one method (the best one) of performing a given operation. The main argument was that otherwise various ways would get mixed up. In the 1970»s an opposite tendency prevailed: students should be encouraged to find their own ways of computation.

However, subsequently in certain books and educational standards the idea has became completely distorted by declaring that at the "basic level" students have to get the result in any way while at the "extended level" they *have* to get it in two different ways. Thus, the previous preference for exploring a variety of ways and permitting children to perform the operation in their own way was replaced by a "must" condition, which shattered the educational sense of it.

Hassler Whitney (1973) gave a vivid description of the weaknesses of the usual way of "pushing mathematics". He also showed specific examples of helping children to learn in their own ways and to think for themselves. One of them is the addition of the type 8 + 5. Children put 20 beads on a piece of wire: ten red beads and then ten blue, say. — Show eight red beads by sliding them to one side! Put a cardboard spacer and slide five more! How many is this together? Children count; the point is that "thirteen" is *their* answer, not teacher's. Let us note that those five are two red and three blue beads. All thirteen form together a configuration 8, 2, 3 of concrete objects; yet, the decomposition 8 + 2 + 3 is not explicit and is not needed to get a correct answer. Gradually the abstract version 8 + 2 + 3 may be perceived by children as a result of reflecting on their actions. From a formal point of view, this is the same decomposition as in (*) above, but didactically there is an enormous chasm between them.

Since 1973 a great deal of fundamental research has been carried out, but the problem of how to encourage independent mathematical thinking in mass education *in an effective way* remains unsolved.

Context of a task, its purpose, and understanding

The significance of concrete objects in the early mathematics education has been repeatedly stressed in the literature. Moreover, they should be *objects to manipulate*; static pictures should not be regarded as a substitute of movable objects (Aebli, 1951). In certain situations, however, it may also be highly important what kind of objects are used.

A celebrated conception introduced by Piaget and Szemińska (1941) is that of *conservation of* (cardinal) *number*. In a standard task the child is first shown a two-part array where the two parts look identical; the equality of cardinal numbers is easy to judge on direct perceptual evidence. The next step is to introduce a mathematically irrelevant transformation which destroys the obviousness of the equality, e.g., elements of one set are spaced out. Finally the experimenter aims to discover whether the child is able to discount the change and maintain the equality (Donaldson, 1982).

Typically, conservation means the child after having seen a set of, say, 10 red counters in a one-to-one correspondence with a second set consisting of 10 blue counters, watch the blue counters be spread out and is then convinced that after the change there are still as many red counters as blue ones; some of them even react: “Of course, there are as many! Why do you ask?”. In contrast, non-conserving children react by saying that there are more items in the set covering a larger area.

Piaget insisted on one-to-one correspondence and did not allow counting. However, even after having counted both sets non-conservers give much the same responses (Gruszczyk-Kolczyńska, 1994). If one considers how counting is grounded in the common practice of adults dealing with questions of equality, a demand of one-to-one correspondence from children would be excessive. The principle of parallelism (saying that the learning process of an individual should follow the order of historic development of human kind) does not apply here!

The conservation of number is attained at the age between 6 and 7 years on average. However, Alina Szemińska (1976–77) modified the standard Piaget test, replacing counters by suitable toys, e.g., by 10 houses without roofs and 10 roofs. Otherwise the method closely followed the original one. The strong semantic component made the new tests much easier. Szemińska used the term *pseudo-conservation* to describe the level of positive responses in such a test. It is quite remarkable that it was attained already by four-year-olds!

What is the main reason *why standard conservation tasks require a much higher level of mental development than those of pseudo-conservation?* A mathematician would find small, flat, round, identical counters as concrete as toys, so what makes the difference? Szemińska explained this by arguing that conservation is a consequence of the *identification of pairs*, which is easier when a correspondence suggests itself, e.g., houses-roofs or handles and baskets without handles.

Yet, another explanation seems more relevant: although the counters are concrete and manipulative, the question of whether there are as many red counters as blue ones does not have any real purpose for the children. It has no reference to their lives and thoughts and has to be understood literally. On the other hand, the purpose of the question whether there are as many houses as roofs is clear: each house must have a roof. Houses without roofs or roofs without houses are useless. Therefore the question whether there are enough roofs for houses makes sense and the child is not deceived by moving the roofs farther apart. The counters, although familiar to the child and concrete, are semantically neutral and give no hint how to answer the conservation question; the child has to decode the linguistic meaning of words “Are there as many ...”.

This example illustrates that *children should perceive for themselves the purpose of a mathematical activity in order to appreciate and understand it.* Also many examples of post-Piagetian interpretations of children’s behaviour reported by Donaldson (1978) confirm this.

The difficulty of many mathematical tasks depends on their contexts and on the language used. A modification of the context and/or the format of a problem, which appears mathematically irrelevant, may drastically change its difficulty for children. Conversely, a mathematical idea may have a multitude of different formulations, at various levels.

Developing critical thinking

Independent mathematical thinking should develop through critical thinking. Children should acquire the habit of checking their computations by themselves and of reflecting on their solutions of real-life problems. This postulate is in marked contrast to the attitude of those teachers who tend to make authoritarian decisions on what is right in students' work and what is wrong.

> While mathematics was proposed to be a subject structured purely by reason, the teaching of mathematics as a global concern developed rigourous structures far removed from any critical enterprise. Instead of being a discipline reflecting critical thinking, mathematics education became associated with domination, control, tests, and rigid forms of communication (Skovsmose and Nielsen, 1996, p. 1259).

The idea of fostering independent mathematical thinking in children is also incompatible with the long-standing fear (widespread in traditional teaching) of exposing a child to any error whatsoever. This fear has been based on the out-of-date empirical theory of reflection (for details, see Aebli, 1951) and also on stimulus-response models of learning. Clearly, if learning consists of a series of carefully devised similar tasks and the child is to imitate the correct way shown by the teacher, then any method appearing to involve possible confusion must be renounced.

This fear has also been influenced by the experience with children learning correct spelling, which is mostly based on visual memory and not on logic; therefore any case of a misspelled word written on the blackboard may have a lasting adverse effect. However, this kind of experience must not be automatically transferred to mathematics. It is not true that a look at 5 + 3 = 9 may later cause the same mistake when the child adds 5 and 3. Moreover, it is advisable to present students with an opportunity to be exposed to certain intentional errors and to detect and overcome them with the teacher's assistance, when the situation is under control.

Students may be given a series of tasks of the form: Which of the following equalities 5 + 3 = 8, 9 – 6 = 2, ... are true? The child may be told to cross out each wrong equality, or to cross out the wrong result and to write down the correct one, or to write YES or NO by each equality. The words such as "true", "untrue", "false", "correct", "incorrect" should come into the children's vocabulary as clear natural words (understood in a context). I would not be afraid of the words "wrong" and "error" provided that they reflect *the child's* opinion (they are criticized by some educators for being politically incorrect, as the phrases "being wrong" and "making errors" are negatively connoted).

The child should have *the right to make errors*. Errors are inherent in learning. Making mistakes by a math learner should be regarded as natural as falls of a ski learner. Errors reveal incompleteness of knowledge; they do not occur randomly and may be rooted in misconceptions, in erroneous beliefs, in an incorrect underlying premise. *The child should not be punished for not understanding something*. However, the attitude of always saying "it's OK", regardless of whether the child's answer is correct or not, is definitely unacceptable. Incorrect results should not be readily tolerated; students should be informed about them or advised to find mistakes themselves, but this must be combined with help and attention so that they can learn from their mistakes. Errors may be springboards for exploration and discussion.

We should also distinguish between a definite error (as in, say, 8 + 5 = 12) and unprecise or vague wording, which normally accompany learning.

Independent thinking should also be developed through critical approach to verbal problems. I believe that students should occasionally be exposed to problems which are intentionally distorted versions of standard well-formulated problems (the distortion should be clear and conspicuous). Such problems are intended to cause a deliberate conflict situation to make students aware of the necessity of checking the reasonableness of the text. There are several types of such problems: problems with *missing* data (incomplete information), problems with *surplus* (extraneous, redundant or not relevant) data, problems with *contradictory* or *impossible* data, absurd problems, "pseudoproportionality" problems, problems with unstated or irrelevant questions (Puchalska and Semadeni, 1987). Their common feature is that they require meaningful, critical interpretation of the text; children should not jump into computations before thinking a little.

Although non-standard problems are absent in traditional teaching, their idea sporadically appears in papers on education. Eighty years ago a leading Polish educator wrote:

> When we formulate problems, it is advisable that we occasionally pose them so as to make children realize that one has to find arithmetical relations between the data. If we present a problem of the form "Sophie is 5 years old, how old is her brother?", then the child has to note that the problem cannot be solved because no relation between the age of Sophie and the age of her brother is given. Posing problems with many irrelevant data [...] we habituate pupils to paying attention to the relations between the data (Jeleńska, 1926, p. 201).

A distinguished American educator has expressed a similar view.

> The objection to so-called "absurd" problems is based upon *a priori* grounds. [...] Part of real expertness in problem solving is the ability to differentiate between the reasonable and the absurd, the logical and the illogical. Instead of being "protected" from error, the child should many times be exposed to error and be encouraged to detect and to demonstrate what was wrong, and why (Brownell, 1942, p. 421, 440).

Markovits, Hershkowitz and Bruckheimer (1984) gave children problems of the type "The height of a 10 year old boy is 140 cm; what will be his height when he is 20 years old". Many students gave absurd answers based on proportionality. Bender (1985) reported similar attitude of children who got the problem: "A postage stamp for a standard letter from Aachen to Munich costs 60 pfennig. The distance from Aachen to Munich is 600 kilometers. The distance from Aachen to Frankfurt is 300 kilometers. How much is the postage for a standard letter from Aachen to Frankfurt?" Many children answered "30 pfennig". Mathematics was viewed as an activity with artificial rules and without any specific relation to out-of-school reality.

In an oft-quoted paper (IREM de Grenoble, 1979), children's responses to the following absurd problem were reported: "There are 26 sheep and 10 goats on a boat. How old is the captain? What do you think of the problem?". Almost 75% of children aged 7 to 9 and 20% of those aged 9 to 11 performed some arithmetical operations on the given numbers without expressing any doubt.

Non-standard problems of various types were included in some Polish textbooks for primary grades (children aged 7 to 10) as extra material, but most teachers skipped them (a likely reason was that dealing with such problems required independent thinking). When they were given, the students were initially surprised and reacted similarly to those in Grenoble. However, usually after the third problem, they grasped the idea, were very active, emotionally involved, and asked for further funny problems of this sort (Puchalska and Semadeni, 1987). The initial confusion may be explained as a result of an unexpected change of the social contract. Indeed, the children knew they had always been supposed to perform some suitable arithmetic operation(s) and it had been tacitly assumed that each word problem had a unique solution. Therefore it is highly important that students getting such problems should be properly introduced to the new convention and have enough time to adjust to it. Presenting them with one single non-standard problem is pointless.

A new dimension to the topic was given by Gruszczyk-Kolczyńska (1986, p.129) who reported on her remedial program for 61 students in grades 1–3 in Katowice. In her efforts to overcome the children's emotional block she used intentionally ill-formulated problems (having contradictory data, say) with success. During conversation with the child she would formulate such a problem and pretend that this was *her* error; her intention was to convince the child that the teacher *can* also make a mistake and to reduce the child's fear of error.

Stimulation without overdemanding

Through astonishment at certain unexpected facts or regularities children may gain an insight into sophisticated mathematical ideas (to paraphrase Aristotle's apt remark "Through astonishment men have begun to philosophise"). Children's surprise: *Why is it so? How is it possible?* opens a way to exploration of such regularities. Szemińska (1991) used the word "amazement" several times in her descriptions of children's efforts to deal with unexpected situations.

A remarkable way of stimulating children's interest has been used by Edyta Gruszczyk-Kolczyńska in her activities with preschoolers in Warsaw. First she tells each child to take ten pebbles (or, say, ten small sticks). When the children are ready, she closes her eyes and covers them with her hands. Then she tells the children to hold some of the ten pebbles hidden in one hand and the remaining pebbles hidden in the other hand. When the children say this has been done, she opens her eyes and comes near to each child successively. She asks the child to show the pebbles in one hand, looks at them, and says how many are hidden in the other hand. Children are amazed. She does not disclose her secret, but asks: "Who knows in what way I find out what is the number of hidden pebbles? If somebody has an idea, please come to me and whisper it in my ear".

Five-year-olds had only two ways of explaining the phenomenon to her: a) you know everything, b) you have magical powers.

In contrast, six-year-olds were able to gain insight into the problem. At the beginning only few of the children were able to find out how many pebbles had been hidden; they became the teacher's "assistants" and replaced her in telling the numbers of pebbles. They conspicuously used their fingers in counting on, but were unable to describe clearly how they got the right numbers. Those children who could not figure out the "secret" did not get any extra hints, but watching peers act was meaningful. After several repetitions, gradually, all the children in the group grasped the idea. It should be noted that no child used subtraction, although many educators had insisted that in such a problem students should use subtraction.

In this way children were given an excellent introduction into what is a mathematical problem. It is significant that such activities face them with one the of most important features of mathematics: computation can help you to find out about something without seeing it. This was also an example of stimulating children to think intensively without requiring anything over their heads.

The maxim: *stimulation without overdemanding* is strikingly illustrated by children's activities devised and described by (Swoboda, 2006).

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Various manipulation functions in solving geometrical tasks

In educational studies of mathematics, the role of manipulation is highlighted. The action is a base for learning early arithmetic. Manipulation in learning geometry is an argumentative topic, because of different theoretical bases for creation of geometrical concepts. Some theories underline a great importance of visual information in forming the first level of understanding geometry. Such approach is present in works of P. Vopěnka or M. Hejný. It result, from our former experiments, that children are able to act in their early years in the geometrical world. Assuming that visual information gives the first stimulus for creation of geometrical concept, we undertook the experiment to observe the role of manipulation in early geometry.

Geometrical concepts as a result of action interiorization?

Piaget's theory of interiorization has a great impact on methods of teaching mathematics in early childhood. It is assumed that child's mathematical concepts emerge by operations and interactions with the real world. An action on the object leads to creation of schemata. As the result, through the process of ***reflective abstraction***, actions can be replaced by symbols and words.

Piaget was in opinion that "it is useful to distinguish empirical abstraction, which draws its information from the objects themselves, from what we call 'reflective abstractions'. The latter proceeds from the subject's actions and operations" (Piaget, Garcia, 1989). This process is important also for geometrical concepts – in his opinion, the child is able to identify the properties of objects by the way in which different kind of actions affect them.

But actions undertaken by children up to 7 years old can lead only to a very limited number of examples that are important for mathematics.

> Experimentation shows that (…) only operations appropriate to class and relational groupings and to the numerical and spatiotemporal structures which resulted from them are used. (…) In the course of stage I (up to about 7–8 years) subjects are most concerned with their practical success or failure without consideration of means. (…) Although the children demonstrate by their behavior that they know how to act in the experimental situation, sometimes successfully, they never internalize their actions as operations, even as concrete operations. (Piaget, Inhelder, 1958).

Taking this point of view, perceptions (sights) have a very small impact on the development of mathematical – logical thinking, including geometrical thinking and the creation of geometrical concepts. The role of actions and manipulations is also problematic. It is not clear to what extend they support the creation of geometrical objects.

A complete acceptation of Piaget's ideas regarding geometry pose many questions. For example, some authors (Clements, at el, 1999, Clements Battista, 1992), criticize the fact, that in Piaget's theory the creation of geometrical concepts starts from topological features.

It seems logical to analyze a relation between Piaget's theory and other theories devoted to creation of geometrical concepts. Apart from other important issues, it is worth tracing the role of perception and the role of manipulation in learning geometry at an early educational stage.

Theories which underline the importance of perception in geometry

Some theories stress the fact that geometrical knowing and understanding is created in a specific way. In those theories, the priority is given to perception, although geometrical „seeing" is not identified with the literal meaning of that word. Geometrical world cannot be perceived directly. It is hidden in the real world, and it is emerging from the surroundings through the special intellectual activity which can be called the geometrical insight (Hejný, Vopěnka).

> The first, and the basic understanding of the real world is the understanding via senses. We look at the world of geometry, but not with our eyes; we learn the world of geometry, but not with ordinary senses. Geometrical seeing is possible only because of the sixth sense. This seeing is not less obvious than seeing the real world using the sense of sight. [...] Who looses the geometrical seeing, can not approach the geometrical world; he can only listen to us, talking about this world. He is as the blind, who finds himself in a gallery and listens, what the others talk about the pictures. (Vopěnka, 1989)

At the beginning, there is no geometrical world nor geometrical object in a child's mind. Only objects from the real world exist. But we focus our attention on those objects in various ways. Sometimes we perceive "something". Vopěnka (1989, p. 19) describes such a situation in the following way: *To see "this", means to focus attention on "this", to distinguish "this" from the whole rest. This, what can absorb the whole attention on itself, we call "phenomenon".* Perceiving "something" creates the <u>first understanding.</u> For example, a child can focus his or her attention on a shape of an object or on a specific position of one object in relation to another. *Phenomena* open the geometrical world to a child. In spite of the fact that our attention is attracted by these phenomena, this first understanding is passive: stimulus goes from the phenomenon. In this depiction, the role of perception is large – the perception of "something" is the first step to creation of the child's own geometrical world.

M. Hejný transforms P. Vopenka's philosophical depiction. He relies upon his own experimental studies and on conclusions derived from Piaget's, Vygotski's and van Hiele's work. In Hejný's theory, the development of understanding of geometrical world goes through various levels. On the first level, there is a possibility to perceive shapes and some relations, but these are (both – shapes and relations) attributes of real objects. A verbal isolation of these phenomena is also possible, by talking about them and calling them. Nevertheless, words such as: *triangle, pyramid,… , long, high,* or skills of making comparisons like: *longer, broader, ….* are still words and concepts related to the real, physical world.

In these depictions, the role of an action is lost. Results of psychological researches confirm that in understanding of shapes, the great importance lays upon the pictorial designate. But the next stage is needed. Acts of perception are important but are not a sufficient source of geometrical cognition. Szemińska (1991, p. 131) states that: *perception give us only static images; through these, we can catch only some states, whereas by actions we can understand what causes them. It also guides us to possibilities of creating dynamic images.*

On the other hand, widely known Piaget's results show that children (on the pre-operational level) have great difficulties in movements reproduction – they are not able to foresee a movement of an object in a space. The process of acquisition of such skills is lengthy and gradual. During manipulations, child's attention should be focused on *action*, not on the very *result of action.* It requires a different type of reflection than the one that accompanied his or her perception.

Experiment

In our experiment, as the basis, we took Vopěnka' and Hejný's theories about the opening of the geometrical world. First of all, we based on the assumption, that the first understanding takes place when a child turns its attention on any geometrical phenomenon. We were interested in situations where children can manipulate. Results of our previous experiments (Swoboda 2006; Swoboda, Synoś 2007) showed that making patterns (arranging them out of blocks, folding out of puzzles, drawing), can fulfil our expectations. Patterns are a friendly environment for children. They are close to their natural, spontaneous activities. Such work gives a chance to connect the process of concept forming with an individual child's actions, which are adapted to his/her own specific activity.

In order to test the possibilities of creating a "path" from perception to manipulation, we prepared an experiment, which took place in March–April 2008. Children from a nursery school, aged 5–6 , were the subject of the series of observations.

Children were tested individually. As a research tool we used "tiles" (two types), shown below (fig.1).

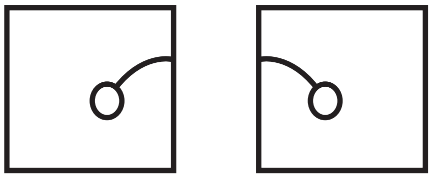

Figure 1. Research tool

Part I.:

A teacher makes a segment of the pattern (fig.2) :

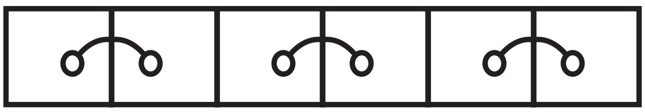

Figure 2. A segment of the pattern prepared by a teacher

On the table, there are also tiles arranged into two separate piles.

Teacher says: Look carefully at this pattern and try to continue it.

If a child doesn't undertake the task, the teacher will say: look how I do it. After that you will continue.

If a child undertakes a task. then after having finished making the pattern, he/she will take part in the next part of an investigation.

Part II.:

Teacher says: Now, please close your eyes, and I will change something in your pattern. After that ,you will say what has been changed. (Teacher exchanges one tile in the pattern, so that the regularity is distorted). Then, the teacher shows the pattern and asks a child: Is there something wrong here? Why? Regardless of the answer received from the child, the teacher says: and now try to correct the mistake I have just done.

Results of the experiment

The importance of visual information

A pattern prepared in our experiment represents an idea of a mirror symmetry (axis symmetry). Axis symmetry functions differently in visual (static) representation than in a dynamic creation of the object's symmetrical image. For a correct pattern continuation, it was necessary to use two different types of tiles, those with the same shape, but oriented differently.

While making the pattern (continuation of the pattern) children started their work by trying to compose one motif using two tiles. This motif was taken as "the whole", which was important for the next work phase. This method was related to the psychological aspect of perception. The holistic understanding was supported by the verbal explanation given by children; when asked: "what does this pattern look like" they answered: *cherries, headphones, if you drew one line at the bottom it would be a car, a tunnel ,a bridge, setting suns.* Just one object arises to multiple ideas. So, these children were working on the pre-conceptual level (according to M. Hejny's theory), their geometrical world was still strongly connected with the real world. The shape was an attribute of the real object.

Some other children worked accordingly to the other strategy; they arranged tiles in a row and took them in turns – one tile from the first pile, one tile from the second pile.

On the basis of these observations, it is hard to say whether children were working accordingly to the idea of a mirror symmetry. Their action and manipulation was stimulated by visual information, which was sometimes supported by a rhythmical movement.

Sometimes children used to wait a long time before they started to manipulate. They were looking at the pattern prepared by a teacher and at the tiles. It took a long time – sometimes one minute or more – until they have analyzed the task, and after that they started their work with a clear, right idea. Therefore, we are in opinion that the main reason for children's work was the visual perception of the regularity.

Example 1. (6 year old boy)

1. T: Look carefully at this pattern and try to continue it.
2. B: *he catches quickly one tile from the left pile, puts it away, takes the second tile from the same pile, gives it back. He looks at the table (6 seconds)*
3. T: You can take it to your hands.
4. B: *He reaches the same pile, takes one tile, gives it back, takes another tile from the bottom of the same pile, gives it back again. 3 second pause. Now he reaches the right pile, takes one tile, leaves it, takes another one and finally decides to place it in the pattern – firstly, from the right side but very quickly changes his mind and puts it on the left side* (see the picture)

Now he works very fast. He continues his work on the left, taking tiles in turns from both piles (second motif). For the third motif, first he takes two different tiles, connects them in hands and only after that connects the whole motif with the pattern. In the meantime, the tiles from both piles mix up on the table and it is not easy to recognize two different sets. The boy stops his work and looks carefully at the table.

5. T: Did you finish?
6. No. *Now he takes one tile from the mass, tries to connect it with the last one in the pattern. On seeing that his choice is not correct, he changes the tile and continues his work, taking successive correct tiles form the table without doubt. In this way he builds a very long pattern, extending it on the right and left side. In the end, there are only three identical tiles on the table. The boy sits motionlessly (18 sec.), looks at the tiles.*
7. T: Do you still want to work?
8. B: No.

This boy spent a lot of time looking at the pattern and at the tiles. He preferred to make a visual analysis than a manipulation – supposedly, the visual information was more important for him. In addition – he knew how to use these information. Perception was the foundation for any his decision, manipulations only supported and verified the undertaken actions.

Children's word argumentations, derived from the second phase of the experiment, also support the visual level. Children experienced great difficulties in explaining why the regularity is destroyed. Their argumentation referred to the observed phenomenon. They tried to explain what *is wrong*, but not *what was changed*. Here are some statements:

Ola: Because there are … two… in the same direction… sides.
Kasia: Because they are the same.
Dominika: Because here it is like this (she shows the "old" tile) and here it is like this (she shows the "new" one).
Michał: Because a tunnel does not exist.

This type of argumentation was supported by indicative gestures. Children used gestures while pointing at the place where the regularity was broken. The importance of such gestures (of pointing at something) is discussed by psychologists: thanks to that gestures, these objects are highlighted. In our observations, children immediately identified the place with the wrong configuration. At this stage, a reflection was related to a figural aspect of the given task.

Such action (pointing) is strongly related to the visual level of understanding and children from our experiment worked at this level. Mental activity was not caused by the physical activity – it was evoked by the transformation of the visual information. A child followed one tile up to another, trying to control the global shape of the whole motif. At this level, all manipulations have the supplementary function. A child was able to detect errors visually, to find irregularities. A child was also able to check visually whether these correct relations exist.

Movement related to the geometrical phenomenon

Other types of manipulations were related to the action of "correcting mistakes" in patterns.

Two different tiles which were used in patterns constituted a symmetrical couple. Children's behavior undoubtedly showed that frequently they were not conscious whether it is and how it is possible to make a motif consisting of two tiles of the same type. The correction of regularities progressed in two different ways:

- A. A child rejected a "wrong tile" immediately and replaced it with the correct tile, taken from the proper pile.

- B. A child started to manipulate the "wrong tile" (despite of his previous experience gathered while he was making the pattern), trying at all costs to obtain the mirror position.

The table below contains the quantitative specification which shows the presence of these strategies in children's work.

Age	Replaced strategy (A)	Manipulative strategy (B)	Helpless
6 year old	10	6	
5 year old	7	7	1

Table 1. Pattern correction strategies

Although convinced that the tiles are in two different types, children sometimes undertook attempts of matching up two tiles of the same type. Such behavior can lead to a conclusion that maybe they were not completely aware of the nonsense of such actions.

Example 2. Martynka, 6 years old (second phase work: correction of mistakes in the pattern)

1. M: A *girl throws away two non-related tiles from the pattern immediately. She looks at the table but no free tiles are on. In this situation, she decides to manipulate these two that she has in her hands: rotates them, attaches one to another in various ways. She does not look at the teacher*.
2. T: So, probably it is impossible to do something with these two tiles.
3. M: *She nods in assent*
4. T: Do you have any idea how to correct this pattern?
5. M: *She does not say anything, still manipulates – one tile she keeps in one position, turns the other one.*

The fact that children undertook manipulations is the foundation for the statement that they tried to put some hypothesis about relations between these tiles. Probably they presumed that thanks to manipulations, they can come across such a position that will give a chance to built the whole motif. In that case, they assumed that an eligible movement leading to a specific, interesting arrangement exists. Such situation is seen it the next example:

Example 3. Karolek, 6 years old (second phase work: pattern mistakes correction)

1. K: *He starts to work immediately. Takes into his hands two identical tiles, which are laying one to another. He leaves one tile without any movement and he puts the second tile in many various positions.*
   After some time, he moves these two tiles closer to him and makes manipulations using only them.
2. T: Keep on trying…
3. K: *He makes various movements on a plane – mainly rotations. At some point, he looks at the reverse sites of the tiles (where no picture was printed): firstly, he looks at one tile, and after some time at the second one.*
4. T: Do you know what have I done? I've changed one tile from these piles (*she points at tiles on the table)*
5. K: *He does not react, he goes on manipulating the tiles for some time. Finally, he takes a casual tile from the table, checks if it fits, takes another one and after the third attempt, he finds the proper tile. Then he makes the whole motif and connects it with the pattern.*

Karolek starts correcting the pattern from manipulating two tiles of the same type. At the beginning he makes movements using only one element. It seems that he has a clear imagination of the results that he wants to obtain. Then, he supports his work by manipulating two tiles at the same time. He sees that the second tile should be "turned" in some way, but he does not know in which. Finally, he displayed a good intuition by looking at the reverse side of the tile. He checked if the picture did not pierce the paper. It is obvious that such experience led him to the following conviction: if he is to have only one-sided tiles, then in order to build a motif, he needs two different types of them. He finalizes his work: patiently, he chooses elements among these lying on the table, looks for those which create a couple with the previous ones.

The shape is accessible to a child in an intuitive manner, and the figure is recognized as the same if its shape is maintained (Williams, Shuard, 1970). Although accordingly to some researcher's opinion the axis symmetry is very difficult for the sight (Demidow, 1989), it makes invariant transformations in the manner which is non-conscious for us. Such an interpretation may not necessarily be contradictive to some Piaget's results (Piaget, Inhelder, 1968) regarding the development of mental images. Therefore, children's actions described in this paper are not yet the base for interiorization in Piaget's sense. They are perceived as a necessary step to obtain the next developmental level. This level is necessary for gathering experience needed for the level, in which the reflection upon the movement would be possible. As it is seen in the table 1, the consciousness of some relations changes accordingly to age. Children know that shapes on the tiles are connected to each other in some way, but they were not quite aware of the relation type. In spite of the fact that they participated in the first stage of experiment during which they gathered experience in building the pattern, they frequently changed the work strategy during the second stage. They tried to force the idea of connecting two one-type tiles, making mainly rotations. By putting the tiles "upside down", they were checking the effect visually. The children's reflection was focused on the result of actions (whether it is possible to fit two tiles) and not on the type of movement leading to success. It seems that there is a huge distance between the children's actions described here, and the understanding of visual dynamic imagines in Szeminska's (1991) sense. In our experiment, manipulations only supported the visual, static information. The rhythm, order and regularity were the factors that inspired children to act and the ones that we controlled. Such model of activity was in accordance with the initial Greek meaning of the word „symmetros", which means „harmonious", „with right proportions". Such feelings were verified visually by the children.

Summary

Observations and conclusions gathered by us should not lead to a conclusion that on the fist level of creation of geometrical concept, children should be deprived of a possibility of manipulations. It should be just the opposite! The perceived geometrical phenomenon should be investigated by means of a spontaneous manipulation. Therefore, the direction should be as follows:

Phenomenon $\rightarrow$ manipulation

At this stage, manipulations are evoked by perception and are subordinated to perception. The manipulation itself is only a tool which enables to reach the aim. While solving the problem, child does not consider what kind of manipulation he/she makes. Thus, it is a didactical abuse to say that all tasks in which a child makes any manipulative activity, lead simply to interiorization of actions. A reflection upon the result of the experiment should not be identified with the experiment itself.

While making conclusions regarding the role of manipulation in creation of geometrical concepts, we are in opinion that, at the level of 5–6 year old children, manipulations can occur in (at least) two different types:

- Deictic motions (through pointing and showing, when a child has a clear idea what he wants to obtain). These are motions that indicate an awareness of relations and connections between particular elements. These are also motions that indicate an awareness of certain disturbances in relations. This awareness is built by visual inquiry of the geometrical phenomenon.

- Manipulation for searching for an effect (I know, what I want to obtain, but I do not know how to reach the objective). A child has a vague feeling that some kind of manipulations can establish an expected relation between objects, but has no idea what kind of movement is needed. Manipulations only support visual imagination.

Second type movements will probably have a great significance for creating concepts of geometrical transformations or dynamic visual imaginations of geometrical objects.

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Is it possible to teach our pupils to think independently?

*To think independently means to think on one's own. People who think independently feel the **need** to make sense of everything based on **personal observations and experiences** rather than on **information they were given without questioning it**. It does not mean that independent thinkers have to invent everything themselves, but it does mean that to think independently they have to trust their own ability to make judgments, even if it contradicts what others say and even if it results in making mistakes.*

Do we need independent thinkers, particularly in the process of teaching-learning mathematics? On the one hand, teaching creativity and independent-thinking are among the goals of teaching, and on the other hand, we have to teach pupils to recognize typical situations and react to them in the most suitable (usually algorithmic) ways. How is it possible to combine both these things? It is very tempting to concentrate just on the memorization of facts and practise algorithmic skills, than to work on creativity and independence in thinking – especially when we don't have much time. But independent thinkers usually bring to our work a new and very often interesting perspective. For example, when pupils have to draw on a chosen piece of paper as many points as possible almost all children did it in the usual way (picture 1), but one boy did it independently (picture 2) – and it was amazing!

Picture 1

Picture 2

So, what we can do to help our pupils develop the **desire** and **ability** to think on their own?

We should start with developing the **desire** to think independently.

First of all, we have to create an environment in which pupils will **feel confident enough to voice their opinion**. It is not easy because very often teachers tend to reward the answers **they** want to hear, and not pay much attention to pupils' thoughts or even discourage those pupils who have different ones.

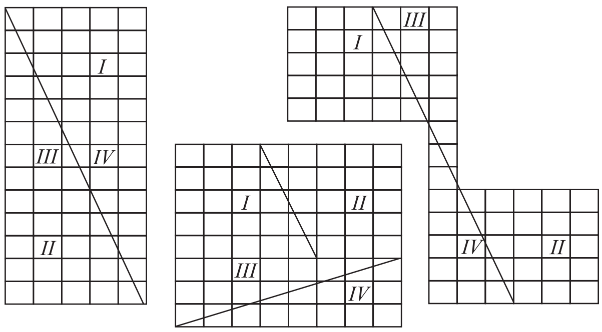
Once we have created a friendly atmosphere, we have to show **the need** to use thinking. For example, instead of just showing a set of tasks and their solutions, we can prepare a situation (cotextualized would be helpful) which introduces a chosen problem. Then, we have to ask questions which the **pupils will think** about it. We should also remember that general questions are in those situations more suitable then direct recall or knowledge questions. And now, we **have to be very patient** and let pupils **express and justify** their opinions. All the time we have to control ourselves to resist **the temptation** to tell pupils what they **should** think. In the end, we should show our pupils the **joy** that comes from being able to think independently and that we are proud of them – rewarding is the best way to motivate pupils and encourage this kind of thinking to happen again. What is more, we have to stop answering the question "why", with "because I said so". Could this kind of answer foster any kind of thinking? The result of this demand for unquestioning obedience is that the pupils stop asking questions, which eventually leads to not thinking.

To develop the **ability** to think independently among our pupils we have to use activity formats similar to the one described above and foster it in almost each situation we face (as often as we can).

How often do we create situations which foster independent thinking? It is awfully time-consuming, and demands a lot of preparation, attention and energy. In addition, the teacher should be able to perceive and appreciate the worth of independent thinking in the pupil and, of course, an independent-thinking teacher can only be an advantage. No wonder that we usually prefer just telling or showing pupils what they should know as opposed to trying to arrange situations in which pupils can find certain knowledge using their independent thinking. For example with small children, with whom we have to share the knowledge of different ways of adding numbers, instead of letting them to invent their own ways, differentiate and group them and lastly to draw conclusions, we just enlighten children with the final product. We can find the same format on every level of learning – for example with multiplying fractions, grouping quadrangles, proving Pythagoras Theorem or finding out the value of geometric series.

In preparing a situation which provokes independent thinking we can also use situations which confused us, for example the "proof" that 65 = 64 = 63 (picture 3).

65 = 64 = 63.



Picture 3

In those situations we can also use critical thinking, which is the ability and willingness to asses claims and make objective judgments on the basis of well-supported reasons, as a tool to think independently.

I would like to finish with the conclusion that although we can't teach our pupils to think independently, we can create situations which provoke independent thought and constantly support them with their struggle which is nicely described in a quotation from E.E. Cummings "A Poet's Advice to Students":

> To be nobody but yourself in a world which is doing its best, night and day, to make us like everybody else – means to fight the hardest battle which any human being can fight; and never stop fighting.

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Problems of formation learning environment

We discuss on major problems and ways of Ukrainian pedagogical innovation which based on democratic transformations in education. Our special study is formation learning community (environment) of successful learners. That modifying methodical frameworks focus on providing student's capacity to be engaged in cooperative learning as a capable individual that know how to initiate and create independent opinions, negotiate and build consensus for problem solving and risk-taking. We present a set of the modern methods and innovative leaning strategies in classrooms that increase the students' capacity to form learning environment as inquiry one. We focus on transforming classroom practices so that they provide a climate of trust, engage students in interesting activity and foster deep inquiry and genuine debate.

The realization of this approach is based on the idea that successful learning activities assume some features of inquiry; a lesson begins to resemble a project with meaningful classroom dialogue and inquiry [1]. It involves taking ideas and examining their implications, exposing them to polite skepticism, balancing them against opposing points of view, constructing supporting belief systems to substantiate them and taking a stand based on those structures. We have convinced that environment stimulates purposeful and productive activities, not traditionally „study work": students are engaged in the practical intellectual work of finding solutions to problems that originate from the real world. It improves the educational process that enables the students to acquire the mathematical knowledge more firmly, to form the practical abilities to use them.

We pay attention to the main peculiarities of learning environment and conditions of one's realization:

- **(i)** <u>goals of education:</u> they reflect the students' hopes based on dialogue (the productive exchange of ideas, attitudes such as tolerance, careful listening to others, assuming responsibility for one's own positions and so on);
- **(ii)** <u>a role of the pedagogue</u>: it reconstructs reality in a problematic form, with the students perceiving and analyzing this reality, the curriculum assumes that students' interests are to be taken into account;
- **(iii)** <u>subject:</u> a teacher (active) and a student (active), <u>object:</u> the entire surrounding world;
- **(iv)** <u>the knowledge is subject to doubt</u>: the doubt must stimulate dialogue, a critical approach and creative activity, education is a creative task;
- **(v)** drastic change of <u>reality</u> according to human needs;
- **(vi)** <u>existence of a problem</u> stimulates the search for its solution.

Three points advance students in their intellectual development [2]:

(i) they should be faced with choices, with materials that invite their comfortable and familiar ways of considering things, get more than one interpretation and the challenge to make and defend their own interpretations;

(ii) they should hear their classmates express points of view different from their own;

(iii) they should be encouraged to reflect, especially in writing, on the ways in which their thinking is changing.

We pay attention to the meta-cognitive processes, i.e. mastering of "thinking strategies", "implementation rules" for cognitive activity. In this context it is essential to overcome stereotypes like "right" and "wrong" responses. Formation the learning environment is impossible without usable knowledge as well as knowledge concerning essence of inquiry and different ways of one. The students create more knowledge and solutions to practice problems, but also to be able to the systematic and efficient habits of idea creation based on the knowledge of the key discipline concepts.

The students know the valuable thing of collaborative/cooperative work and the improvement of their argumentation during conducting research and presenting results to others. In the learning environment the students are attracted to listening to different opinions of their classmates and creating an atmosphere that support free acceptance of the another ideas or argumentative rejection them, tolerant and interdependent social behavior.

In the formation of learning environment the teacher's role is facilitating students, that realized by three objective: to design contexts that promote one's inquire for thinking; to develop strategies for encouraging thinking skills; to develop strategies, especially portfolio, for authentic assessment of thinking as a mean of evaluating work. The main results of that approach is changing students' role, that realized in self-searching activities and grouping forces (instead of remembering ready-made knowledge). Moreover it makes available shifting their attention to the individual peculiarities and uniqueness that provides possibilities of solving educational problems as individually as in group; encouraging their interests to following and training methods of inquiry (investigating, experimenting, interviewing, surveying, writing, and so on).

There are some work questions [2]: What is the main question posed by this piece? What answer does it offer? What reasons are offered in support of that answer? What evidence is offered in support of each reason? What reasons or facts are left out – things that might have supported different answer to the question? What "facts" are we expected to accept on faith? What nominal assumptions are made? What value assumptions are made?

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Suggestions regarding the support of independent thinking in mathematics

I. Changes and trends in arithmetics and early algebra

Part 2

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Scheme – oriented educational strategy in mathematic

Scheme is understood as a memory structure that incorporates clusters of information relevant for comprehension. It gets embedded in a person's mind by a repeated "stay" in a certain kind of environment (one's house, school, shopping centre). Scheme–oriented mathematical education is described and illustrated on a primary level. This paper surveys the experience with the implementation of this teaching method in teacher's training.

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1. Introduction

The aim of the contribution is to analyse and discuss one cognitive phenomenon which, to our opinion, might be used to improve contemporary education of mathematics (not only) in elementary schools.

Contemporary educational strategy of mathematics in the majority of our (Czech and Slovak) elementary classes is topic oriented. It means that each time, the whole period of mathematical lesson is focused on a particular topic: counting, sharing, measuring, etc. The alternative educational strategy presented in this paper is scheme oriented. It means that mathematical lesson is focused on solving problems, in which more mathematical schemes are addressed at the same time. A set of what we call mathematical environment is an educational tool for such an approach.

2. Scheme

When someone asks you about the number of windows or lamps in your flat or house, probably you will not be able to give an immediate answer. However, after a little while you will answer the question with absolute certainty. You will imagine yourself walking from one room to another and counting the objects that you were asked about. Both of the required pieces of information and many other data about your dwelling is embedded in your consciousness, as a part of the scheme of your flat. We use schemes to recognize not only our dwellings, but also our village, our relatives, interpersonal relationships at our workplace, etc.

Specialized literature gives various connotations of the term 'schemes'. The following quote by R.J. Gerrig provides a rather loose definition that serves our purposes. "Theorists have coined the term schemes to refer to the memory structure that incorporate clusters of information relevant to comprehension… A primary insight to scheme theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units." (Gerrig, 1991, pp. 244–245).

A scheme-oriented education is based on creating two kinds of schemes:

semantic schemes rooted in everyday life experiences of a pupil and *structural* schemes which are 'pure mathematical' and have no direct linkage to pupil's life experience.

Structural schemes of early mathematics are created within different semantic schemes and after introducing a structural language of ciphers, they start to shake off this semantic supervision.

As an example, let us consider the concept of 'number 3' as one of the ten basic elements of the early mathematic structural scheme. This mathematical concept originates from the semantic scheme of rhymes, creating the ability to produce the rhythm, which synchronizes words and movements. A child's speech *one, two, three* is accompanied by handling obejcts. After the performance, its last word *three* must be repeated to point at the product of the process of counting: the set of three objects. Once the synchronization is created, a child is able to count objects. Both abilities – handling objects and synchronization – are nested in everyday life experiences. Word *three* in both its appearances starts to create a concept of 'three objects' as the first pre-structural concept of the concept 'number 3'. The first word 'three' brings the processual and the second 'three' the conceptual understanding.

The described pre-structural concept of 'number 3' is not completed yet. So far it is supported only by one semantical scheme and three others have to be added: number as an address, number as an operator of comparison and number as an operator of change (see below).

3. Semantic schemes breeding up the early arithmetic structural scheme

As mentioned above, there are four different semantic schemes in which a number appears: status S (number, magnitude), address A (in terms of place or time; the temporal address can be either linear of cyclical), the operator of change Ch and the operator of comparison Co. The symbol O will stand for the operator, when there is no need to specify its particular type.

In some cases there is no sharp boundary between these schemes. Take for example this situation: Ann (who stays at the second floor) has to go three floors up in order to see Betty (who stays at the fifth floor). Here number 3 can be regarded either as an operator of change (Ann moves) or as an operator of comparison (Betty stays 3 floors above Ann).

Numbers are the soil of an early arithmetic scheme. The core of it is an operation – addition and subtraction. Here the variety of semantic types comprise at least eight issues:

S + S = S	3 female and 5 male pupils, 8 pupils in total.
S – S = S	If 5 out of 8 pupils are boys, then the remaining 3 are girls.
S ± Co = S	E has 3 pets. F has 1 pet more/fewer than E. Thus F has 4/2 pets.
A ± Co = A	J. is 8 years old. R. is 1 year older/younger. R. is 9/7 years old.
S ± Ch = S	Eve had 5 €. Today she received/lost 2 €. Now she has 7/3 €.
A ± Ch = A	Cid used to live on the 5th floor. He moved 2 floors up/down. Now he lives on 7th /3rd floor.
Co ± Co = = Co	Eva read 5 pages more than Fay, who read 2 pages more/fewer than Guy. Eva read 7/3 pages more than Guy.
Ch ± Ch = = Ch	The number of bus-passengers increased by 7 persons at the first stop. At the second stop it increased/decreased by 5 persons. At these two stops the number increased by 12/2 bus-passengers.

Table 1

The key semantic model, mastery of which is the decisive step towards understanding Early Arithmetic scheme, can be written as ±O ± O = O. Our longterm experience substantiated by the experimental research of Ruppeldtová (2003), clearly indicate that the problems of using only operators are among the most demanding problems for pupils. We would like to know why operators are so demanding?

Commentary 1. The answer to the given question is rooted in different perceptions of statuses and addresses on one hand, and the operators on the other. The status and address are both enclosed data. Information such as “there are 5 chairs around the table” does not generate any further questions concerning numbers.

The operator is, by contrast, an example of an open data. The information “there are two chairs fewer” provokes the question such as: ‘what was the original number of chairs?’ and ‘how many chairs are there now?’ These two numbers are *virtually* present in the operator of change. The accuracy of the above thesis is confirmed by the behaviour of pupils who are assigned to such operator problems. When given such a problem, they keep asking for virtual data and for explanations as to how to deal with them. These pupils clearly did not have enough experience with numerical situations that feature exclusively the operator of change. That is why the current situation might be improved by incorporating operator tasks already in first-grade primary school curricula. In order to achieve this goal, we elaborated several environments. Three of them are presented in this paper.

4. ‘Walk’ environment

The teacher (and later one of the pupils) gives an order and another pupil(s) walks accordingly to it. Sample commands: 1. Three steps forward, go! 2. Two steps, then one step, forward, go! 3. Three steps forward, then two steps backwards, then one step forward, go! To keep steps of pupils equal, there is a set of about a dozen marks on the floor of the class. After this warm-up stage, the addition is introduced by the following scene: Two pupils, C and D, are standing side by side. Pupil C receives the following command: Three steps forward, then two steps forward, go! Pupil D receives the command: Five steps forward, go! Both pupils, C and D, eventually end up standing side by side again. The entire scene is accompanied by words and body movements, and can be classified as a walk representation of the addition 2 + 3 = 5.

The problem originates by concealing one of the three numbers. The given situation, therefore, leads to three problems: 2 + 3 = ?, 2 + ? = 5, ? + 3 = 5. The concealed number here has been replaced by a question mark. In the class scenario it is replaced by the word ‘what?’; e.g. problem written here as 2 + ? = 5 will be presented as

Problem 1. Pupils C and D are standing side by side. Pupil C goes 5 steps forward. Then a teacher says ‘Pupil D two steps forward, then *what?* steps forward, go!’.

The class already knows that it is necessary to replace the word ‘what?’ by a suitable number. In the given case the number is ‘three’.

‘Walk’ environment brings a natural possibility to introduce the pre-concept of negative numbers (which is impossible within the environments dealing only with a status). Negative numbers are represented by backward steps. The experiment proved that even firstgraders can easily solve the problem 2 – 3 = ? Pupil C receives the following command: Two steps forward, then three steps backward, go! Few pupils immediately and the whole class after a while found the solution as a command for pupil D: one step backward. In such a way a concept of negative number starts within the backward movement.

It is necessary to stress that on this stage, there is no numerical notation for negative numbers. Symbols like '-1' will be introduced later, not before the fourth grade. At that time, in each of our experimental classes, pupils used a sign minus as a natural description for both: addresses of places/years below the zero and operators of change in decreasing directions.

Commentary 2. The 'Walk' environment allows pupils to build their semantic schemes, from which four Early Arithmetic fundamental sub-schemes emerge: number ordering, addition, subtraction within natural numbers and pre-concept of negative number. The most important in this environment is a great support for understanding of addition and subtraction of operators, particularly the operator of change.

5. 'Footprint' environment

So far we dealed with short commands only. When a longer command with five or even more numbers appears, it will be difficult for a pupil to remember it. Thus, there is a need to find a way how to record a long command. Pupils start to create their own recording systems using fingers, dots, lines,… Finally one or more of pupils finds an arrow as a suitable tool for recording steps. A teacher now can take this pupils' discovery as a common language for describing commands and walk performances. It is important that no authority such as a teacher or a textbook actually brought this new language. Pupils found it themselves and therefore it is their own language. In such a way the Footprint environment is introduced[1] .

The arrow representation of the addition 2 + 3 = 5 is given by Figure 1.

[→ → | → → →] = [→ → → → →]

Figure 1

On Figures 2a, 2b, 2c there is an arrow representation of tasks 2 + 3 = ?, 2 + ? = 5, and ? + 3 = = 5 respectively.

[→ → | → → →] = [ ] , [→ → |] = [→ → → → →] , [| → → →] = [→ → → → →]

Figure 2a Figure 2b Figure 2c

There are two substantial differences between environments: Walk and Footprints. The first one is due to the fact that the Walk is ephemeral, while the Footprints is permanent. Words and steps will fade away, but Figure 1 will remain.

The second difference resides in the fact that the permanent language allows to create more demanding tasks than the language of word commands. This can be illustrated by one problem dedicated for fourgraders.

Problem 2. Fill in the three empty boxes with six arrows to fulfill both equations:

[| ← ← ←] = [→ |] = [ ] . (1)

Remark. Only arrows of the same direction are allowed in each box.

[1] J. Slezaková (2008) performed a number of experiments with the sole aim of finding appropriate graphemes for this model. In the end, arrow symbols were chosen as the most appropriate for children of 6 to 8 years of age.

Commentary 3. Having translated problem 2 into algebraic notation, it can be written as the system of three equations:

$x - 3 = y + 1 = z$, $|x| + |y| + |z| = 6$;

number x is positive if arrows in the first empty box in (1) are oriented right (→) and negative if these are oriented left (←). The same procedure is valid for letters x and z.

Within this environment even such a difficult rule

minus out of minus makes plus (2)

can be presented as a walking performace by means of the command 'turn about' abbreviated by TA. For example the expression 3 – (2 – 4) can be produced as shown in arrow languge:

→ → → | TA | → → | ← ← ← ← | TA

Figure 3

Our experience with this interpretation of the rule is very positive. Many of our pre-service teachers, future elementary teachers declare this performance to be ‚the proof of the rule'.

Commentary 4. While solving various Walk & Footprints problems, a pupil gets familiar with this double-environment and develops his/her mathematical understandig in different areas: ordering, addition, and subtraction of whole numbers; later on also solving system of equations, pre-concept of the absolute value of a number and even several ideas from probability and statistics.

6. A 'bus' environment

The bus route is marked by several (shall we say five) stops in the classroom, which we shall label A, B, C, D, and E. The stops are at particular places within the classroom, e.g. the teacher's desk, a washbasin, map, whiteboard, wardrobe, the piano, … A cardboard box stands for the bus and plastic bottles stand for the passengers. The bus departs from the initial stop A and ends up in the terminus E and anyone can get off and get on each stop. The decision-making is done by the pupils who act as conductors at individual stops. All the pupils see how the passengers are getting on and off, but only the driver can see the inside of the bus (the box); the driver is the pupil who is carrying the box. When the bus has reached the terminus, the teacher asks the pupils how many passengers they think there are in the bus. Each pupil writes his/her tip into a table and then checks it by looking inside the box.

The pupils first try to remember the number of passengers, later they start to keep written records. After the fifth or sixth round of the game, the teacher asks whether anybody remembers how many people got off at stop B. The teacher asks such questions during every subsequent performance, which forces the pupils to invent a more resourceful way of recording the entire process.

Story. In one experimental class, where each stop had a distinct colour, the teacher, after a tenth round asked: "What happened at the green stop? Did the number of passengers increase or decrease? By how many?" Only several pupils understood those difficult questions. One pupil immediately gave a correct answer. Then he explained to his classmates the secret of his solution.

Prior to the performance, the boy drew up 5 oval shapes in five different colours which matched the colours of the bus stops. Each oval stood for a bus standing at the appropriate stop. Then, using arrows, he recorded the performance. Two arrows directing the green oval represented two passengers entering the bus at the green stop and four arrows directing out of the oval represented four passengers getting out of the bus. Having used this record, the boy immediately saw that at the green stop the number of passengers decreased by two.

After several performances, some pupil came up with a method of table- recording. On request of the teacher, the discoverer showed the record to the class and the teacher started writing his/her own performance records on the blackboard. The discovery was not made in all experimental classrooms, in some of these the teacher had to clue the pupils and gave them the table record. The pupils eventually used the record found in the upper half of table 2. The table shows us that 2 passengers got off at stop B, while 3 got on; on the stop C one passenger got off and 4 got on. After a month, the table was complemented with a row entitled "go" which stands for the number of passengers on the bus between individual stops. E.g. Table 2 indicates that the highest number of people – 7 – went from stop C to stop D.

	A	B	C	D	E
out		2	1	4	5
in	3	3	4	2	
go	3	4	7	5	

Table 2

The teacher gives the pupils tasks that reside in blotting out some of the numbers. It can be illustrated by a rather demanding task assigned to 4^{th}-graders.

Problem 3. Design a performance table if you know that the same number of people got on at each of the A, B, C, D stops. 5 people got off at stop D and the maximum of 5 people and the minimum of one person got off on the subsequent stops.

Commentary 5.The Bus environment was tested by seven different teachers in 4 first-grade and 3 second-grade classrooms. The environment was given a very warm welcome, especially by pupils. This influenced even those teachers who did not trust the environment at first. The environment uses the experience of pupils with bus route.

After illustrating three semantical environments, we turn our attention back to the starting concept of our research, to the scheme.

7. Theory of generic models – a tool for understanding mathematical scheme

Remark. From now on, under the term 'scheme' we mean 'mathematical scheme'.

In chapter 2 we gave Gerrig's definition of the concept scheme. Then, we gave several illustrations of mathematical schemes. However, the concept still remains in a theoretical level. So far, we have no clear idea how to use this concept in our educational praxis. Namely we do not know

a) How to evaluate the quality of particular mathematical scheme in a given pupil's mind? and (3)

b) How to help this pupil to overcome possible developmental obstacles?

The goal of this chapter is to find such a tool. To do this, we will use our theory of generic models, which we briefly (i.e. without illustrations) describe[2] here.

[2]The theory designed by the author's father, Vít Hejný was first published in 1977 in the Slovak language. Its first English presentation is found in the article Hejný (1988), and its current version is in the paper Hejný, Littler (2006).

Our model of the process of gaining knowledge is based on stages. It starts with motivation and its cores are two mental lifts: the first (generalisation) leads from a concrete knowledge to a generic knowledge and the second (abstraction) from a generic to an abstract knowledge. The permanent part of the knowledge gaining process is crystallisation – inserting new knowledge into the already existing mathematical structure.

The whole process can be depicted in the following scheme consisting of two consequent levels:

motivation → isolated models → *generalisation* → generic model(s)

generic model(s) → *abstraction* → abstract knowledge → crystallisation

As we see, the generic model, the pivot between experiences and abstract knowledge, plays a decisive role.

Motivation. We see motivation as the tension which occurs in a person's mind as a result of the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between 'I do not know' and 'I need to know', or 'I cannot do that' and 'I want to be able to do that'. Sometimes this discrepancy comes from other needs too.

Isolated models. First experiences of a new piece of knowledge come into mind gradually and have a long-term perspective. For instance, the concepts of fraction, negative number, straight line, congruency or limit develop over many years at a preparatory level. For more complex knowledge, the stage of isolated models can be divided into four sub-stages:

1. The first concrete experience – the first isolated model appears and this is a *source* of new knowledge.
2. A gradual 'collecting' of more isolated models, which at this stage are separate.
3. Some models begin to refer to each other and create a *group*. The feeling develops that these models are 'the same, in a sense.
4. Finding out the reason for the 'sameness', or even better, the correspondence between any two models. These models create a *community.*

The above sub-stages can be useful for us when we investigate how a new idea gradually develops in a pupil's mind. It often happens that a new sub-stage, not presented here, appears and that one of those presented does not appear at all.

The stage of isolated models ends with the creation of the community of isolated models. In the future, other isolated models will come to a pupil's mind, but they will not influence the birth of the generic model. They will only differentiate more detail in it.

Generalisation and generic model(s). In the scheme of the process of gaining knowledge, the generic model is placed over the isolated models indicating its greater universality. The generic model is created from the community of its isolated models and has two basic relationships to this community:

1. it denotes both the core of this *community* and the core of *relationships* between individual models and
2. it is an example or representative of all its isolated models.

The first relationship denotes the construction of the generic model; the second denotes the way the model works.

Abstraction and abstract knowledge. The generic model remains an object representative and does not allow for a higher level of structuring acquired knowledge. Therefore, the next step of knowledge development must be abstraction, i.e. disconnection from an object characteristic of a generic model. This shift is accompanied by a change of language and an object representative is exchanged for a symbolic representative. The symbolic representative brings about higher abstract understanding of the knowledge or knowledge area in question than the previous object representative does. This process is intellectually demanding and requires a lot of time and effort from a pupil. The abstract knowledge is only rarely a consequence of AHA-effect, i.e. a sudden sight of truth. A majority of abstraction processes run in small stages. Creating abstract knowledge is based on the assumption that the symbolic representative is autonomously constructed or at least interiorized by an individual. If the symbolic representative is implemented in a pupil's mind from the outside in a ready-made form it usually only stays on a memory level as 'the knowledge without understanding'.

Crystallisation. After its entrance into the cognitive structure, a new piece of knowledge begins to look for relationships with the existing knowledge. When it discovers disharmonies, the need arises to remove them by adapting the new knowledge to the previous knowledge and, at the same time, to change the previous knowledge to match the new knowledge.

The above description of crystallisation is imprecise in two aspects: first, it suggests the image that crystallisation only begins when the abstract knowledge has been constructed. Second, it supposes that the only thing that is added to the cognitive structure and takes part in the process of crystallisation is the abstract piece of knowledge. Neither is true. Each new mental step, which plays a role in creating the new abstract knowledge, immediately becomes a part of the whole cognitive structure and plays a role in crystallisation. None of the pieces of knowledge which a pupil constructs has a final form and each is being polished, changed and broadened all the time. This permanent development of knowledge is a typical sign of the quality of non-mechanical knowledge.

8. Cognitive mechanism of the birth and the rise of mathematical scheme

Now we are prepared to answer questions (3). In brief we can say that

a) the quality of particular mathematical scheme in a given pupil's mind can be evaluated accordingly to the set of its generic models and a web by which these models are connected;
b) the most frequent developmental obstacles originate from the lack of generic models and their connections; thus the way of overcoming these obstacles is to build these missing models and connections.

In more details we describe the birth of the scheme and its internal organization.

Firstly, we clarify the birth of scheme. Isolated models and clusters of these models provide a breeding ground for a scheme. A scheme only appears with the origination of the first generic model. A child may discover that the total of 2 footballs and 3 footballs equals 5 footballs, the total sum of 2 and 3 dolls is 5 dolls, but s/he has not yet developed a scheme for adding small numbers. This scheme is only developed once the child has discovered that these calculations can be done by counting on fingers, which thus become a generic model for adding small numbers.

Secondly, we underline that scheme is a dynamic organisation of heterogeneous elements. The word *organisation* emphasises the fact that it is not just a set of elements, but also a set of bonds between these elements. The adjective *dynamic* refers to both short- and long-term mutability of the set of elements and of the entire organisation. Schemes may be either more stable or more flexible. Some flexible schemes originate by the amalgamation of smaller schemes. E.g. the scheme of the term "rational number" was created by amalgamating the schemes of the terms "fraction" and "negative number". The dynamism of a scheme is shaped by an internal conflict following the introduction of a new isolated model: a 1st-grader discovers that one half is a number, or a 4th-grader realises that a quadrilateral can be non-convex, or an 8th grader finds out that there can be a triangle with indefinitely large circumference and indefinitely small area.

9. Conclusions

As stated in the Introduction, one of the goals of this paper is to prepare our future activity while working with teachers. The key role in this work will be played by schemes, isolated and generic models. Here, these concepts are illustrated mostly in the area of arithmetic. However, in this work we will deal with geometry as well. As a base for the geometrical activity we will use ideas described in Swoboda (2006), Jirotková (2007), and Hejný, Jirotková (2006).

The concept of the scheme is elaborated in several theories. For example in a famous study of Gray, Tall (1994), in which the concept of procept is introduced. We read:

"The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as a part of the *wider mental scheme*. Symbolism that inherently represents the amalgam of process/concept ambiguity we call a 'procept'" … (p. 116).

The concept of the scheme is also incorporated in the APOS theory. It presupposes "… that mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them in *schemes* to make sense of the situations and solve the problems. … Finally, a *scheme* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemes which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept" (Dubinsky, McDonald, 1999).

Interpretations of the scheme in procept theory and in APOS theory are similar to our interpretation. The comparison of these theories can be found in Hejný (in print)

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Perceptions of numbers of 5 to 6 year old children

In this article we deal with observation of children's perceptions of number. We investigate process in which to numerical information the conceptions of numbers are assigned. The numerical information is word three and the child's drawing is used to mediate the numerical conceptions of children. The experiment was realized with the children in the kindergarten. We analysed drawn children's conceptions of number 3 and created concept map from those drawings.

Theoretical basis of the surveyed problem

We concentrate on the observation of children's perceptions of number in our research. When investigating the exteriorization of the processes of children's thinking about numbers, we will distinguish two processes. The process, in which to different conceptions of numbers (C) the numerical information (I) is assigned: ; and the process, in which to numerical information the conceptions of numbers are assigned: . The numerical information is such information (a word, move, gesture …) that contains at least one number. If we want to take a deep look into the primary conceptions of numbers of children, the information created in the first phases of mental process and closely connected to his/her private world of numbers are very important. We analyze the process in our experiment, whereby the child's drawing is used to mediate the numerical conceptions of children.

According to Piaget (1970), the drawing is a form of semiotic function, which has its place between the symbolic game and figurative conception.

R. Davido (2001) claims about the child's drawing that a child probably does not already know or does not already want to express himself/herself verbally; however, his/her drawings can reveal much about him/her and his/her real and imaginary world. The quality of not only motoric but also of cognitive and emotional growth of a child is reflected exactly in the child's drawing. It is the drawing that helps to organize the world into the unity of shapes.

That is the reason why we have chosen drawing as an appropriate entrance gateway into the inner world of children.

The publication written by Hejný, Stehlíková (1999), in which the authors analyze the process of the emergence of the world of numbers from the world of things, provided the theoretical basis for the realization of our research. The introduction of the world of numbers Hejný realizes in accordance with Popper's idea of three worlds. He classifies the world of numbers as a part of the second and the third world that Penrose (1994) entitled as the mental world and the world of culture. Hejný understands the world of numbers as the structure with the essence in person's semantic conception of a number. The conception consists of three components: the number, its attachment to the world of things, and the knowledge of an individual, in which the number and its attachment are situated. The world of things (the first world according to Popper) interprets

Hejný as the set of all the human's conceptions of things, events, situations and relations existing within the perceived world. The world of numbers originates within the world of things in the course of the intellectual growth of children.

The process of the emergence of the world of numbers from the world of things has two components:

- Verbal: the emergence concerns the words -number words. The child acquires them as the sounds whose meaning he/she understands only vaguely. Words three, four, five are intuitively grouped together such as words white, blue, green
- Semantic: the emergence concerns the meaning of the numerals. This component is crucial for the construction of the world of numbers. The process of construction is divided into four developmental stages:
  1. The stage of opening of the world of numbers – the child begins to distinguish between the singular and the plural, i.e. between one and lots of.
  2. The stage of separated conceptions – the child already has the conception of what three balls means or what three fingers means, but perceives them in isolation. He/She already does not know that these conceptions represent the same amount of things.
  3. The stage of universal conceptions – the child knows that individual conceptions of numbers may stand one for another. Fingers or counters of an abacus are becoming universal models for the child. The achievement of this stage means the construction of the world of numbers that is associated with the world of things.
  4. The stage of abstract conceptions – the child is already able to manipulate with the conception of "three", "four", etc. meaningfully. He/She does not need to frame this conception within the world of things. The world of numbers gained independence.

There are various classifications of conceptions of natural numbers into classes and subclasses. In our paper, we will sort the children's conceptions into the following groups:

1. the natural number as a cardinal number, i.e. for counting (the number of dots on the face of a cube, the number of fingers on the hand, three pears, five cars, …)
2. the natural number as an ordinal number, i.e. for sequencing (the first in the finish, the third floor, …)
3. the natural number for identifying (the class 3.A; the bus No. 7; its 12 a' clock; …)

Within the experiment, we were working with children attending kindergarten. Therefore, we mention some skills and knowledge about numbers that children are supposed to know at the age of 5 and 6 (according to the educational activities realized in kindergarten):

- to recite the numbers from 1 to 6 in correct order,
- to create a group of objects with a stated number of elements (bring one scarf, pass me two papers, put aside two dolls, take one plate, …)
- to determine the number of objects in a given set (How many cars are there?)
  - by estimation (dots on the die, fingers on the hand),
  - by counting the one by one,
- to express the number of objects of a set with the use of fingers, dot symbols, respectively to assign the number symbol to a given group.

The experiment

The aims of the experiment

For the realization of the experiment we chose number 3, because it is the number from the numerical scope 1–6 and it is freely distributed in fairy tales, advertisements and other areas of ordinary life.

The following aims were stated:

1. to obtain drawn children's conceptions of numbers created according to the provided numerical information in the process (I → C),
2. to classify created number conceptions into stated groups and to analyze in which phase of formation of semantic component of the world of numbers do these conceptions appear,
3. to compare the number conceptions of children before and after the intervention of the experimentalist,
4. to create a concept map from children's drawings,
5. with the help of a video recording, to evaluate the behaviour and activity of children in the experimental activities.

The course of the experiment

The experiment was realized in January 2008 with the group of 20 children in the age of 5 to 6 years in the kindergarten in Nábrežie Mládeže Street in Nitra between 8 a.m. and 10 a.m. The lesson procedure was recorded on video. The experiment was divided into four phases. The first three phases covered the interaction between the experimenter and the children, the fourth one between the children and their teacher:

1. The first phase passed without our intervention. The children were asked to draw what they imagine when we say number 3. The process (I → C) was monitored, in which the numerical information was the word 3.
2. On the basis of the reactions of children we told them at the beginning of the second phase several examples from real life where they might come across the number (talk between experimenter and children):

   E: Has anybody seen something with number three in advertisements?
   Ch: Markiza (TV)
   E: Is Markiza with three? Where is number three there?
   Ch: Because you just press three on the remote control and that is Markiza.
   E: I live on the third floor. Is there anybody else living on the third floor?
   Ch: I live on the tenth. And I on the third.
   E: When you use an elevator, what do you press?
   Ch: Three.

   A chocolate bar 3Bit is shown to children.

   E: And what is this?
   Ch: Chocolate bar 3Bit.
   E: Do you know it? The three again!
   E: Adam was on the football tournament. Do you know what their rank in the order was?
   Ch: What does it mean “the order“?
   E: That somebody is the first, second, third.

We asked them to draw their imagination of number 3 again. We approached to children individually and talked with them about their picture and tried to help the weaker ones. Children presented their drawings. We observed how their conception of the number changed after our intervention.

3. In the third phase of the experiment, all the pictures were fixed to the board. Children were asked to arrange the pictures into groups according to some similarity and to explain their choice. Thus the concept map of children concerning their images of number three were created. At the end of the third phase the children were asked to look for the number three in their domestic environment.

4. The fourth phase passed without the presence of experimenter. The next day the teacher made a record about the reactions of children on this topic.

The evaluation of the experiment

The whole activity took 60 minutes. All children participated actively in the activity; they were concentrated for 35 minutes. After this time had passed, 10 children were able to concentrate on the next activity. Children that up to now are not able to concentrate for longer time on one activity kept running away from our activity. Two of them are diagnosed as ADHD.

The evaluation of the first and the second phase of the experiment

The pictures of children were arranged into the groups according to their conceptions of numbers. Children figured number 3 as the quantity or as an identifier. Six children produced rich pictures, however, without any connection to number 3; one child figured only the shape of number 3 that did not express any conception of the number.

After the dialogue was held with children about number 3 and its various forms around us, the children drew pictures related to number 3 again. In comparison with the first illustration, in the new portrayals number 3 appeared represented as an identifier more times, and for the first time as the order.

The illustration of number 3	The first phase Number of children	The second phase Number of children
Quantity	12	13
Order	0	2
Identifier	1	4
Shape of the number	1	1
No model	6	3

Table 1

Comparison of pictures from the first and the second phase

- three children among those six children who did not assign any concept to number 3 in the first phase could not react correctly even in the second phase. Two children produced models of quantity and one child depicted number 3 as the identifier. These models did not reproduce the conceptions of number 3 introduced by us. Children that did not create any model seemed in general to be very weak in other educational activities, too.
- Five children expanded their primal conception of number of new models as follows:

Number of children	The first phase → The second phase
2	quantity → identifier
1	identifier → quantity
2	quantity → order

Table 2

The enhancement of children's conceptions of numbers was influenced by our preceding dialogue.

Two children created interesting conceptions, which we have labelled “3 in 3”, i.e. in one conception of number 3 another conception of this number appears – three tables with three drawers, three cubes with three dots, three numbers 3 …

The first phase	The second phase
	

Figure 1. Figure of one child

The number conceptions of one child from the first and the second phase of the experiment are depicted in the Figure 1. There are many conceptions of number 3 in both pictures. The enrichment of number conceptions occurred in the second phase – the child did not depict number 3 only as the quantity, but also as the order – the third on the podium. The models “3 in 3” can be found in both phases. Following the richness of drawings, we can conclude that the child is aware of the fact that the same amount of different objects represents the same number, i.e. the child has reached the stage of *universal conceptions*. Similarly various pictures were produced by another four children. Other twelve children were able to assign to numerical information *I* only one or two conceptions of number 3. Thus it can be concluded that they so far do not realize that the same amount of different things represents the same number, they perceive them in isolation – they have reached only the stage of *separated conceptions.*

The evaluation of the third phase of the experiment

Each child presented his/her picture and fixed it to the board. After that, the children were asked to group those pictures that they think have something in common. Thus the groups of three trees, three flowers … were created. It can be seen that children grouped only the pictures depicting the same amount of the same things. They also created a group of two pictures depicting train, which, however, did not represent any conception of number 3. It can be concluded that children’s conceptions are at this age strongly fixed to the world of things.

We have produced the following concept map of number 3 together with children (Figure 2).

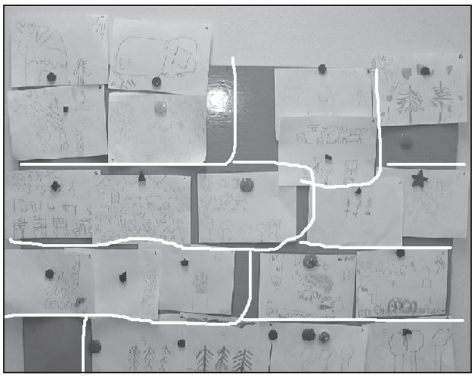

Figure 2. Concept map

The evaluation of the fourth phase of the experiment

The next day, all children, except for one that did not take part in the experiment, reacted to the appeal of the class mistress. Children created miscellaneous number conceptions, in which they presented number 3 as the quantity. These new concepts were independent from the previous shared activity and they manifested deep interconnection with the inner world of children. Children expressed the amount of number 3 with those objects from their lives, to which they have the emotional attitude.

Conclusion

Educational activities aimed at improvement of mathematical conceptions in nursery schools embody mainly in the process, ergo in assigning the numerical information to different number conceptions. In the experiment, the task of the children was to assign to given numerical information the different number conceptions. We were monitoring the process children do not commonly come across. The overall activity of children, as well as the richness and diversity of their drawings confirm that the preschool age children are playful and spontaneous in expressing their conceptions and feelings. The drawing proved to be a very appropriate form of expression, because even a withdrawn girl refusing to communicate with us verbally expressed herself in this way. Even the children unable to connect their conceptions with the world of numbers were drawing; moreover, they were able to react appropriately the next day, too. This confirms that our creative activity had particular influence on children of this age, even if they seemed to be passive at that time. The created concept map points to the ability of children to group the same number conceptions. Within the process of shared sorting of pictures, the children expressed themselves on the level of separated conceptions of numbers within the semantic component of the process of the emergence of the world of numbers from the world of things. Particular pictures are, however, the evidence of the fact that some children have already reached the stage of universal conceptions. Presented experimental activity was enhancing and interesting for the children, as they were able to react appropriately upon this topic and to look for another "threes" in their surrounding spontaneously even later on. We hope to pursue the improvement of number conceptions and the construction of the world of numbers in our further research on the chosen sample.

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Primary school pupils' miscoceptions in number

A group of four universities from the UK, Czech Republic, Israel and Italy collaborated to find common misconceptions in number across the four countries. The paper cites some of the misconceptions founding all the countries and looks in detail at a specific task which the authors developed together with the results gained from the task over the whole primary range of pupils and beyond.

Introduction

Over the past two years researchers in four universities in the United Kingdom, the Czech Republic, Israel and Italy have been looking into the misconceptions which primary school pupils have in mathematics. What are the major misconceptions? How do they arise and what can we do to eliminate them? Are specific misconceptions only found in one country or are some of them found across the four countries involved in the project?

These were some of the questions we asked ourselves at the first meeting. We also were of the same opinion that mistakes/misconceptions should not been seen by the teacher and the pupil to be avoided at all costs and that some form of punishment should be given for making mistakes. We firmly believe that mistakes can be used as a great educational tool to bring about understanding of the underlying concepts behind a particular piece of mathematics.

We clarified in our own minds the difference between mistakes and misconceptions. A mistake often occurs because a pupil is trying to work too quickly and thus miscopies, makes a simple mistake in a calculation or makes an error remembering one of the many facts which s/he has had to commit to memory such as 7 x 8 = 54! Where as a misconception, as the name suggests means that the pupil does not understand the concepts on which the particular mathematical topic is based. For instance a pupil who has been told that when you multiply by 10 you 'add a zero to the right hand side of the number you are multiplying', when asked the cost of 10 pens which cost Ł1.35 each gives the answer Ł1.350. In other words the pupil does not know why s/he should add a zero in the first place probably because he got the knowledge second-hand either from the teacher or a peer. The important point being that the 'rule' gave the correct answers for the level of mathematics the pupil was doing at the time but was never told that the rule only applied to natural numbers.

Background

Many books and papers have been written on misconceptions in primary mathematics but few if any have looked at the underlying reasons for these, for instance Gelman and Gallistel, (1978), Hughes, (1986). Several books have been written on specific problems which arise from children's misunderstanding of concepts such as the equality sign and zero, (Haylock, Cockburn, 2008 and Lakoff, Núńez, 2000 on matters relating to zero and Jones, 2006, relating to equality). Pupils use the equality sign in several ways, first and foremost as the completion of a process such as 3 + 4 = 7 to which they use the words '3 add four is 7'. They seldom see it as an equivalence as in 3 + 4 = 1 + 6 or as in the case, when as far as the pupils are concerned the question is set the wrong way round, 6 = 2 + ?, I am sure we have all seen the pupil who strings equality signs together which do not make sense but still the pupil gets the right answer, for instance $32 + 56 = (30 + 2) + (50 + 6) = (30 + 50) = 80 = (2 + 6) = 8 = 80 + 8 = 88$.

Similarly in our research we have found many instances of pupils not understanding that zero has a quantity. They only have the idea that it is a place holder. Later in the paper we will give examples of students crossing out the digit 1 rather than zero when they say they are crossing out the 'smallest' digit. This was found not only with primary school pupils but also with pupils up to Grade 10.

Obviously the problems with zero are also linked to problems with place value. Our research supports the view expressed by Ashlock (2002) that pupils are taught the value of the digits 1 to 9 and the concept of the value of the places in the denary system, that is if you ask a young child to bring 4 sweets from then dish then this can be done, and if you point to a particular place they can probably tell you 'those are tens' but these two concepts are rarely put together so that the pupil can recognise that the value of the 2's in 203 and 472 are very different. A feature of modern life has contributed to this. Rarely do you hear a pupil say the number 125 as one hundred and twenty five, it is more usually read as one, two, five! Hence the pupils are not recognising the value of the digit in its place setting.

When dealing with decimals in the upper primary school we found exactly the same misconceptions which had been found in the long-term UK research by the APU (1975–80,1985) For instance when the number of digits after the decimal point in one number is different from the number in the second then this causes problems eg. To the sum 5.07–1.3 three types of misconception arose giving the answers: 4.4, 4.04, 4.94. In the first answer the pupil ignored the zero in the first number and treated the seven hundredths as seven tenths. In the second the three was put under the 7, the decimal points under each other and the units under each other, is this a case of zero meaning 'nothing'? The third answer was derived from putting the 1.3 under the 0.07 ignoring the decimal point in the second number and then subtracting. These results all came from one class.

Another area which high-lighted many misconceptions were tasks dealing with the number line which was unmarked apart from the end points. Values near the endpoints were generally approximately true but in over 95% of the pupils work there was no idea of proportionality when putting numbers on the line. For instance when Grade 2 pupils were asked to put 0, 5 and 8 on a number line showing the position of 1 and 10 most of the answers given had all the numbers between 1 and 5 on the line. Many older pupils showed decimals of the form 0.25 as being below zero.

Experiment

We obviously could extend the misconceptions we found considerably but the authors had specific tasks to test across the four countries. At one of the first meetings of the team, various misconceptions which the members had met were discussed and some ten tasks devised which it was hoped would show whether the misconceptions were found in just one of the countries or across all of them. The experimental design was the same for all the tasks in all the countries. We involved teachers we knew and with whom we had worked before, presenting them with the tasks which were carefully aimed at specific age-groups or syllabus development and asked them to administer the tasks on our behalf. Before they gave the pupils the tasks several in-service sessions were given at which the tasks were discussed, what we hoped they would show and all the tasks were attempted by the teachers so that they were clear what was required.

In this paper we want to concentrate on one task for which we devised four levels. The basic task which was set was:

Given an n-digit number, (n being dependent on the age/ability of the pupil), strike out a digit so that what is left is the largest possible (n –1)-digit number. The digits in the original number must not be reordered. Starting again with the original number, strike out a digit to make the smallest possible (n – 1) digit number. Thus 'n' in the case of 594 would be 3 with the largest 2-digit (i.e. n–1) number result being 94 and the smallest 54.

The numbers we gave the four levels were determined by the number syllabuses in the four countries involved. If pupils were working on two digit numbers we gave them three digits in the task, for them to cross out one as so reduce the problem to a two digit number which should have been within the pupils' competence. This methodology was continued through the levels.

Level I. (Knowledge 0 to 20) 213, 120.

Level II. (Knowledge 0 to 100) 2109, 892.

Level III. (Knowledge >1000) 23015, 15023.

Level IV. 352091, 432502.

We developed this task because our experience in schools suggested that some pupils had 'rules/strategies' in their memories which they applied to determine largest and smallest numbers. The ones we had met were:

- Strategy 1: Cross out the 'smallest' digit to get the largest number;
- Strategy 2: Cross out the 'largest' digit to get the smallest number;
- Strategy 3: Cross out the right hand digit to get the largest number;
- Strategy 4: Cross out the left-hand digit to get the smallest number;
- Strategy 5: Cross out the zero to get the largest number.

Table 1. Strategies met in school prior to research

Hence our objectives when designing the task were to:

(i) give us insight into the pupils' knowledge and understanding of place value. So, for example, in the case of 213 do children consider *place value* (as we would hope!) or opt for crossing out the 'largest' digit (i.e. 3) when endeavouring to create the smallest 2-digit number?

(ii) see if there were common misconceptions across the four countries of the project. If so, what could we learn, for example, about the different teaching methods used?

(iii) analyse the misconceptions to determine their origin. This included looking for patterns in the children's responses to see if, for instance, they responded correctly to all the tasks they were given excepting those involving zero(s).

(iv) how early in their school life did these strategies occur?

(v) provide ideas for re-education/education to eradicate the misconception(s).

To give an example of the range of possible answers we could expect if all the strategies listed above were used by a class, we have tabulated the possible results for level 1 task, using the numbers 213 and 120

Number	**Correct solution**	**Smallest/largest digit – strategies 1 and 2**		**Right/left digit – strategies 3 and 4**		**Zero, largest smallest – strategy 5**	
213							
Smallest	**13**	21		13			
Largest	**23**		23		21		
120							
Smallest	**10**	10		20		12	
Largest	**20**		12		12		12

Table 2. Possible answers to level 1 task using strategies above

Note: for the number 120, if the pupil uses the right/left digit strategy then the 'smaller' number determined is bigger than the determined 'larger' number. This actually happened in several cases and would suggest that these pupils had a strategy in their minds which they were convinced would give them the correct answer, so they never checked to see whether the results they gave were sensible or not.

Results

Analysing three classes of 58, 6 to 7 year-old pupils who worked with the numbers 213 and 120 the following facts emerged:

For **213**

Largest number

Cross out the smallest digit (correct answer) 27 pupils
Cross out the right-hand digit (units) 10 pupils
Cross out the left-hand digit 12 pupils

Smallest number

Cross out the left-hand digit (correct answer) 24 pupils
Cross out the largest digit 7 pupils

Less than half the 58 pupils got the two parts of the task correct. There were 9 pupils who used the twin strategies 'cross out the left-hand/right-hand digits for the smallest/largest numbers respectively'. No pupil who used crossing out the smallest digit to get the biggest number used cross out the biggest digit to get the smallest number.

In one of the three groups the pupils must have had some instruction of how to find the smallest number for 213 since the whole group gave the same incorrect solution, cross out the '3' and then reversed the remaining digits '21' to get the number 12. Their solutions for the other tasks were not significantly different from the other two groups.

For **120**

Largest number

Cross out the left-hand digit (correct answer) 42 pupils
Cross out the smallest (zero, right-hand) digit 12 pupils

Smallest number

Cross out the biggest digit (correct answer) 33 pupils
Cross out the right-hand (zero) digit 19 pupils

Eleven pupils used the twin strategies 'cross out the left and right-hand to get the smallest and largest numbers respectively'. Zero caused many pupils difficulties since many are not sure of its function. They probably have been told that 'zero is a place holder' or that 3 – 3 'is nothing' which is then written as '0'. Another phenomenon connected with zero arose with older pupils.

Looking across the three groups different phenomena occurred in each group. In one group the most common misconception for finding smallest number for 213 was crossing out the '3'– the largest digit. Another group had different misconceptions for finding the largest number in 213, in fact contradictory misconceptions since the same number of pupils crossed out the right digit as crossed out the left one. In this class more pupils crossed out the right digit zero than got the correct answer. A number of pupils re-arranged the digits if their strategy did not seem to give the expected result, even though they were told at the start of the tasks not to alter the order of the digits.

Thus we did not have to look very far to see how early these inappropriate strategies were used. Responses to the level 2 task, showed that all the strategies used in level 1 appeared again at this level with crossing out of the largest and smallest digits being the dominant misconception of these pupils whose ages ranged from 8 to 10 years. A new phenomenon appeared with these pupils. Some pupils who said they were crossing out the smallest digit to get the largest number in 2109 and 9120 crossed out the '1' and not the zero getting 209 and 920 respectively. This gives the incorrect answer for the first number and a correct answer for the wrong reason in the second. We considered that this was evidence that these pupils did not see zero as a digit having a value but purely as a place holder in the denary place value structure.

At level 3 exactly the same misconceptions occurred with those classes working with five digit numbers as occurred earlier. That the misconceptions are perpetuated is worrying since it means that these pupils have not been given tasks which will help to identify these misconceptions or possibly the teachers have marked an answer correct which was obtained by incorrect reasoning.

We gave the level 4 task to secondary as well as primary school pupils and even with pupils as old 14 to 15 years the same misconceptions as were found in Grade 1 were apparent. Many grade 6 pupils used the twin strategies 'cross out the right hand digit to get the biggest number and the left-hand digit to get the smallest number' for 352091, getting 35209 and 52091 for the biggest and smallest numbers respectively! Even at this age there were some pupils who crossed out the '1' rather than the zero.

Conclusions

The important results of our analysis were:

- the strategies listed earlier in the chapter were prevalent in Grade 1 and were found in every grade up to 10;
- most pupils were inconsistent in the strategies they used to solve the problems both across tasks and within tasks. This would suggest that these pupils considered each task individually. Not many pupils used both of the twin methodologies – smallest/largest digit or right/left-hand digit to solve one task;
- pupils did not check to see if their answers were sensible;
- many pupils did not connect the cardinal value of the digit with the place value where it was situated;
- very few pupils took a number and crossed out the digits in turn to determine smallest and largest.

Teachers need to give their pupils tasks which will high-light misconceptions and these particular misconceptions are particularly hard to eradicate because they sometimes give the right answer for the wrong reason and in many cases have been used throughout their school career.

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Focusing on children's early ideas of fractions

This paper describes children's understanding of quantities represented by fractions in quotient, part-whole and operator situations. The studies involve two samples of first-grade children, aged 6 and 7 years from Braga, Portugal. These children were not taught about fractions before. Two questions were addressed: (1) How do children understand the equivalence of fractions in quotient, part-whole and operator situations? (2) How do they master the ordering of fractions in these situations? Quantitative analysis showed that the situations in which the concept of fractions is used affected children's understanding of the quantities represented by fractions; their performance in quotient situations was better than their performance in the other situations.

This paper aims to describe part of a research project focused on the effects of situations on children's understanding of the concept of fraction.

Independent thinking is only possible with understanding. To improve children's understanding of a mathematical concept one needs to know how the concept develops. According to the Vergnaud's (1997) theory, to study and understand how mathematical concepts develop in children's minds through their experience in school and outside school, one must consider a concept as depending on three sets: a set of situations that make the concept useful and meaningful; a set of operational invariants used to deal with these situations; and a set of representations (symbolic, linguistic, graphical, etc.) used to represent invariants, situations and procedures. Following this theory, this paper describes studies on children's informal knowledge of quantities represented by fractions, focused on the effects of situations on children's understanding of the concept of fraction.

Different classifications of situations that might offer a fruitful analysis of the concept of fractions are distinguished in the literature. (Kieren, 1988, 1993) distinguished four types of situations – measure (which includes part-whole), quotient, ratio and operator – referred by the author as 'subconstructs' of rational number, considering a construct a collection of various elements of knowing; Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Marshall (1993) distinguished five situations – part-whole, quotient, measures, operator, and ratio – based on the notion of 'schema' characterized as a network of knowledge about an event. More recently, Nunes, Bryant, Pretzlik, Evans, Wade and Bell (2004), based on the meaning of numbers in each situation, distinguished four situations – part-whole, quotient, operator and intensive quantities. In spite of the diversity, part-whole, quotient and operator situations are common to these classifications. These were the three situations selected to be included in the studies reported here.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, 2/4 in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In a quotient situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). In operator situations, the denominator indicates the number of equal groups into which a set was divided and the numerator is the number of groups taken (Nunes et al., 2004). In an operator situation, if a boy is given 2/4 of 12 marbles, means that the 12 marbles are organized into 4 groups (of 3 marbles each) and the boy receives 6 marbles – that is 2 groups of the 4 into which the 12 marbles were organized. Thus number meanings differ across situations. Do these differences affect children's understanding of fractions when building on their informal knowledge?

Applying Vergnaud's (1997) theory to the understanding of fractions, one also needs to consider a set of operational invariants that can be used in these situations. Extending Piaget's analysis of natural numbers to fractions, one has to ask how children come to understand the logic of classes and the system of the asymmetrical relations that define fractions. How do children come to understand that there are classes of equivalent fractions – 1/3, 2/6, 3/9, etc – and that these classes can be ordered – 1/3 > 1/4 > 1/5 etc? (Nunes et al., 2004). It is relevant to know under what condition children understand these relations between numerator, denominator and the quantity. The invariants analysed here are equivalence and ordering of the magnitude of fractions, more specifically, the inverse relation between the quotient and the magnitude.

Thus these studies considers a set of situations (quotient, part-whole, operator), a set of operational invariants (equivalence, ordering of fractional quantities), using linguistic combined with pictorial representation. In this paper we investigate whether the situation in which the concept of fractions is used influences children's performance in problem solving tasks. The studies were carried out with first-grade children who had not been taught about fractions in school. Two specific questions were investigated: (1) How do children understand the equivalence of fractions in part-whole, quotient and operator situations? (2) How do they master the ordering of fractions in these situations?

Previous research (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) on children's understanding of division on sharing situations has shown that children aged 6 and 7 understand that, the larger the number of recipients, the smaller the part that each one receives, being able to order the values of the quotient. However, this studies were carried out with divisions were the dividend was larger than the divisor. It is necessary to see whether the children will still understand the inverse relation between the divisor and the quotient when the result of the division would be a fraction. The equivalent insight using part-whole situations – the larger the number of parts into which a whole was cut, the smaller the size of the parts (Behr, Wachsmuth, Post & Lesh, 1984) – has not been documented in children of these age. Regarding equivalence in quotient situations, Empson (1999) found some evidence for children's use of ratios with concrete materials when children aged 6 and 7 years solved equivalence problems. In part-whole situations, Piaget, Inhelder and Szeminska (1960) found that children of this age level understand equivalence between the sum of all the parts and the whole and some of the slightly older children could understand the equivalence between parts, 1/2 and 2/4, if 2/4 was obtained by subdividing 1/2. Concerning operator situations, previous research on children's informal knowledge (Empson, 1999) shows that children aged 6 and 7 found it difficult to understand the operator concept.

Although some research has dealt with part-whole, quotient and operator situations with young children, these were not conceived to establish systematic and controlled comparisons between the situations. There have been no comparisons between the three situations in research on children's understanding of fractions. Research about the impact of each of these situations on the learning of fractions is difficult to find. We still do not know much about the effects of each of these situations on children's understanding of fractions. This paper provides such evidence.

Methods

Participants

In a first study, Portuguese first-grade children (N = 80), aged 6 and 7 years, from the city of Braga, in Portugal, were assigned randomly to work in part-whole situations or quotient situations with the restriction that the same number of children in each level was assigned to each condition in each of the schools. In a second study, another group of Portuguese first-grade children (N = 40), aged 6 and 7 years, from the same two schools were working in operator situations with the same restriction for each level in each of the schools. The children had not been taught about fractions in school, although the words 'metade' (half) and 'um-quarto' (a quarter) may have been familiar in other social settings.

The tasks

An example of each type of task presented to the children is given below (Table 1). The instructions were presented orally; the children worked on booklets which contained drawings that illustrated the situations described. The children were seen individually by an experimenter, a native Portuguese speaker.

Problem	Situation	Example
Equivalence	Part-whole	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar in 2 equal parts and eats 1 of them; Ann breaks hers into 4 equal parts and eats 2 of them. Does Bill eat more, the same, or less than Ann? Why do you think so?
	Quotient	Group A, formed by 2 children have to share 1 bar of chocolate fairly; group B, comprising of 4 children have to share 2 chocolates fairly. Do the children in group A eat the same, more, or less than the children in group B? Why do you think so?
	Operator	Anna and Martha each have a bag with 4 marbles. Anna splits hers into 2 equal groups and puts 1 group in her red bag. Martha splits hers into 4 equal bags and decides to put 2 groups in her blue bag. Does the red bag have more marbles than the blue bag? Does the blue bag have more marbles than the red one, or do they have the same number of marbles? Why do you think so?
Ordering	Part-whole	Bill and Ann each have a bar of chocolate the same size; Bill breaks his bar into 2 equal parts and eats 1 of them; Ann breaks hers into 3 equal parts and eats 1 of them. Who eats more, Bill or Ann? Why do you think so?

Problem	Situation	Example
Ordering	Quotient	Group A, formed by 2 children has to share 1 bar of chocolate fairly; group B which consists of 3 children has to share 1 chocolate fairly. Who eats more, the children of group A, or the children of group B? Why do you think so?
	Operator	Eve and Ruth each have a bag with 6 lollypops. Eve splits hers into 2 equal groups and puts 1 group in her red bag to eat later. Ruth splits hers into 3 equal bags and decides to put 1 group in her blue bag. Does the red bag have more lollypops than the blue bag? Does the blue bag have more lollypops than the red one, or do they have the same number of lollypops? Why do you think so?

Table 1. Types of problem presented to the children in each type of situation

Design

In both studies, the six equivalence items and the six ordering items were presented in a block in random ordered at the beginning of the session. The numerical values were controlled for across situations.

Results

Study 1

Descriptive statistics for the performances on the tasks for quotient and part-whole situation are presented in Table 2.

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	2.1 (1.5)	2.95 (1.54)	0.6 (0.7)	0.6 (0.5)
Ordering	3.3 (2.1)	4.25 (1.3)	1.45 (1.4)	1.2 (0.83)

Table 2. Mean (out of 6) and standard deviation (in brackets) of children's correct responses by task and situation

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and problem solving situation (quotient *vs* part-whole) as between-participants factor, and tasks (Equivalence, Ordering) as within-participants factor.

There was a significant tasks effect, ($F(1,76) = 18.54$, $p<.001$), indicating that children's performance on ordering tasks was better than in equivalence tasks. There was a significant main effect of the problem situation, ($F(1,76) = 146.26$, $p < .001$), and a significant main effect of age, ($F(1,76) = 4.84$, $p < .05$); there was a significant interaction of age by problem solving situation, ($F(1,76) = 7.56$, $p <.05$). The older children performed better than the younger ones in quotient situations; in part-whole situations there was no age effect. There were no other significant effects.

An analysis of children's arguments was carried out and took into account all the productions, including drawings and verbalizations. Table 3 shows the frequency of children's arguments and the rate of correct responses for problems in quotient and part-whole situations. Children presented more valid arguments based on the inverse relation between the number of recipients and the size of the shares, when solving problems in quotient situations. In part-whole situations, the valid arguments were based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken. In part-whole situations the most frequent arguments used when were based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut.

	Type of situation							
	Quotient (N = 240)				Part-whole (N = 240)			
	Equivalence		Ordering		Equivalence		Ordering	
Type of argument	Freq.	Prop.	Freq.	Prop.	Freq.	Prop.	Freq.	Prop.
Invalid	17	0	17	.01	10	.01	6	.02
Perceptual comparisons	46	.03	50	.09	–	–	–	–
Valid	88	.27	94	.38	14	.03	15	.06
Only to the dividend (numerator)	76	.09	64	.14	172	.18	177	.13
Only to the divisor (denominator)	13	.03	15	.01	44	.05	42	.01

Table 3. Frequency of arguments type and proportion of correct responses when solving the tasks in quotient and parte-whole situations

Study 2

	Problem Situation			
	Part-whole (N = 40; mean age 6.9 years)		Operator (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	0.6 (0.7)	0.6 (0.5)	2 (1.9)	2.6 (1.9)
Ordering	1.45 (1.4)	1.2 (0.83)	2.6 (2.2)	2.7 (2.1)

Table 4. Mean (out of 6) and standard deviation (in brackets) of children's correct responses by task and situation

Descriptive statistics for the performances on the tasks for part-whole and operator situation are presented in Table 4.

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7--year-olds) and problem solving situation (operator *vs* part-whole) as between-participants factor, and tasks (Equivalence, Ordering) as within-participants factor. There was a significant mains effect of tasks ($F(1,76) = 15.23$, $p < .001$), indicating that the children's performance in ordering was better than in equivalence problems. There was a significant main effect of the situations, ($F(1,76) = 22$, $p < .001$), indicating that the children's performance was better in operator than in part-whole situations. There was no significant age effect and no significant interactions.

	Operator Situation (N = 240)			
	Equivalence		Ordering	
Type of argument	Freq.	Prop.	Freq.	Prop.
Invalid	2	0	1	0
Based on number of units	229	.36	231	.42
Valid	4	0	4	0
Only to the dividend (numerator)	5	.02	4	.02

Table 5. Frequency of arguments type and proportion of correct responses when solving the tasks in operator situations

Children's success on solving problems in operator situations relies on the comparison of the number of units, ignoring the existence of groups and judging the number of units by counting. The inverse relation between the divisor and the quotient is lost in operator situations.

Discussion and conclusion

The situations in which fractions are used have an effect on children's understanding of fractions. Children's ability to solve problems of equivalence and ordering of quantities represented by fractions is better in quotient than in other situations. The levels of success in children's performance in quotient situations, supports the idea that children have some informal knowledge about equivalence and ordering of quantities represented by fractions. These results extend those obtained by Kornilaki and Nunes (2005), who showed that children aged 6 and 7 years succeeded on ordering problems, in sharing situations, where the dividend was larger than the divisor. These results showed that the children still be able to use the same inverse reasoning when dealing with quantities represented by fractions. The findings of these studies also extended those of Empson (1999) who showed that 6-7-year-olds children could solve equivalence and ordering problems in quotient situations, after being taught about equal sharing strategies. The children of these studies were not taught about any strategies. These findings suggest that children possess an informal knowledge of fractions that can be successfully explored using quotient situations. If it is so, why should we keep introducing the concept of fraction to children using part-whole situations? Maybe we should explore more about which is the best situation to introduce children to fractions in the classroom. What sequence of situations should be explored in the classroom to offer a better support to children's independent thinking about fractions?

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Connections – as a fundamental element to constructing mathematical knowledge (exemplify of one pre-algebra task)

Polish primary school do not talk about algebra a lot. Teaching mathematics on this level of education is oriented on arithmetic. In my paper, I would like to present a part of results from my research carried out in a primary school among fourth grade students. This research concerns discovering the regularity, which leads to algebraic reflection.

Introduction

During the last few years, much time has been spent on discussion about learning algebra in a primary school. We say about an "early algebra" and algebraic thinking. (Mutschler, 2005). It is very hard to separate an algebraic thinking from an arithmetic one. One of the ways of developing algebraic thinking is a "superstructure" of arithmetic thinking. It is made by means of generalizing arithmetic contemplations through un-changing constants. Tasks that concern discovering of arithmetic-geometric dependences, which it is necessary to generalize and write by symbols, serve this purpose. It is very hard for the primary school's student to move to symbolic notation. First of all, the student says the general relation and next s/he tries to notice it by symbols. In order to understand algebraic language, the student has to start from understanding its basic component – the letter (Turnau 1990).

Discovering and perceiving regularity by students is an important problem present in the world trends in teaching mathematics. In many countries, teaching mathematics is closely connected with the rhythm and the regularity. We can find references to the description of researches concerning discovering and generalization of noticed rules (Zazkis, Liljedahl, 2002; Littler, Benson, 2005).

Searching for a regularity is an extremely effective method while solving mathematical problems; it is a strategy of solving tasks. As E. Swoboda (2006) says:

> … perceiving a regularity is a desirable skill. Activities, during which the child has to perceive the regularity and act accordingly to the rule are stimulating his or her mental development. These activities are the basis of mathematical thinking for each level of mathematical competence.

The child learns/develops its mathematical knowledge through building its own cognitive structures, a web of interrelationships, mental "maps" (Hejný, 2004, Skemp, 1979). Accumulated experience enables to create a so-called data set, used by a child to build up its mathematical knowledge. This inner structure of mathematical knowledge (internal mathematical structure – IMS, Hejny 2004) is – as prof Hejny says:

> ... dynamic web of connections with many elements of knowledge, such as concepts, facts, relations, examples, strategy of solutions, algorithm, procedures, hypothesis, ..., creating nodes of this web. All this cause the existence of IMS. IMS is a web by itself, connecting these all elements. At the same time IMS is the way of organizing all these elements which create knowledge.

The essential factors which help a child to develop its mathematical knowledge are interactions with its environment, particularly during the teaching-learning process (e.g. during mathematics classes). It is present during a teacher-student interaction and a student-student interaction as well. The best way to activate these interactions is a group work in cooperating teams.

The appearance of a reflection is very important. Reflection on our experience is a perfect starting point for understanding the world (constructivism). Everyone creates his own 'rules' and mental models, which we try to apply in order to understand and use our experience of mastering the knowledge of our environment. The reflection appears when we have to manifest our ideas. While expressing our thoughts, we look for an appropriate form of words (Wygotski, 1989). Reflection does not appear automatically among 7–11 children. Therefore, a conversation during a cooperation with students is an opportunity to recognize their mental processes while solving their tasks

The aim of research

I have been dealing with a perception of regularities and appliance of discovered rules by students for some time. Presented investigation is a part of a series of research concerning the perception of regularities by students on different levels (Pytlak, 2006, 2007). The results of Polish students from PISA test and one of PISA's task (called "Apple trees" – Białecki, Blumsztajn, Cyngot., 2003) were an inspiration to take on this subject.

The aim of my research was to get answers for following questions:

- Will 9–10-years old students be able to perceive mathematical regularity and, if yes – in which way do they "think" about regularities and what is their thinking processes while solving tasks in which they have to discover and use noticing rules?
- Will they be able to cooperate while solving the task?
- To what degree this common work will have an effect on the way of solving the task and discovering the regularity as well as using them in the task?

Presented research was carried out in February 2008 among students from a fourth grade of primary school. The research contained four following meetings, during which students were solving following tasks. All meetings were recorded by a video camera. After the research, the report was made. Students worked in pairs. Researcher talked with every group of students while solving tasks by them.

Twelve students from fourth grade of primary school took part in this research (9–10 years old children). Students had work sheets, matches (black sticks), ball pens and a calculator. Before students started their work, they had been informed that they can solve this task in any way they would recognize as suitable; their work would not be graded; teacher would be videotaping their work and they can write everything on the work sheet which they recognize as important. The research material consists of work sheets filled by students, as well as of a film recording of their work and a stenographic record from it.

The research tool consisted of four sheets and each of them consisted of two tasks.. Tasks were as following: students make a match pattern consisting of geometrical figures – once there are triangles and another time there are squares with a side length of one match. In the first two sheets the figures were arranged separately, in the second two – connected in one row. The question was: How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 of such figures? Results should be written in the table. In the task two, there was a question about a number of matches which are needed to construct 10, 25 and 161 of such figures (Littler 2006).

Patterns, which were the subject of next tasks, were as following:

1. Separate triangles
2. Separate squares
3. Connected squares
4. Connected triangles.

Choice of tasks and the order of sheets were not random. The problem was to check if students will benefit from their earlier experience while solving new tasks; will once elaborated solving strategy be applicable during the next task.

This task and the way of its presentation (four following sessions) were something new for students. Up to this time, during math lessons they did not solve tasks concerning the perception of the appearing rules and generalization of noticed regularities. It was a new challenge for them.

The results

The first two sheets students solved very quickly. They did not need to construct the pattern consisting of proper figures, they already started to fill in the table and next, they answered the question two. They were perfectly able to give the rule according to which the pattern was constructed. Some of students arranged only one figure (one triangle in case of the first sheet and one square in case of the second sheet). It was rather a kind of marking what kind of figure was applied in the task than supporting the task solving process.

Difficulties appeared during the work with the third sheet. The first difficulty concerned the expression: “connected in one row”. Next obstacle appeared while shifting from the table to the question two. Students did not have enough sticks in order to continue arranging the pattern. Besides, in the table they gave consecutive values and in the question two a “jump” appeared. At the beginning, the problem for students was to fill this gap. In order to give the answer, they started to analyze the previous solution of the task and the way of constructing the pattern.

Solution strategies were as following:

The first sheet – filled automatically; the discovered rule is: add three to a previous value (relate to the table) or multiply the number of triangles by three. All answers and formulated rules were correct, students were able to make a generalization, they did not use any symbolic notation.

The second sheet – students noticed an analogy to the previous task (from the first sheet). Some of them applied the rule “multiply by three”... Solving this task lasted less time than in the case of the first sheet.

The third sheet was a challenge for students. At the beginning, they were trying to transfer a solving method from previous sheets. But seeing that it is ineffective, they looked for another solution. They started to analyze contents of the task. Next, they arranged, using matches, a fragment of a pattern – for two, three squares. They discovered the rule: the first square made of four elements, each following of three elements. Therefore, in order to give the number of needed matches, one should add three to the previous number. After filling the table, two ways of actions appeared: continuation of “adding numbers three” to ten squares or searching for “components” in the table, using previously obtained data. Initially, students were convinced of correctness of their method. Only a conversation with the teacher, as well as verification of the applied method for data from the table (i.e. will it be as such for the number seven) caused the change in the way of thinking and discovering the proper dependence.

The fourth sheet was also a challenge for the students. Here however, they used their own experience gained while working on the third sheet, so the solving process of the task progressed quite efficiently.

I would like to look closer at the solution of the task from the third sheet, which was made by two girls: Sylwia and Nicola.

Sylwia and Nicola's work

Nicola is a girl who copes with school mathematics very well and operates with knowledge on the abstraction level (that is available on her level). Sylwia is a much worse student who needed a visual representation to solve the task. While solving tasks from two previous sheets, the girls did not work together. Sylwia worked with Paulina. Before they started to solve the task, Sylwia arranged a few triangles, and next she used it to fill the table together with Paulina.

As a justification for their activity concerning the table, they gave the rule: every time I add three. Moved to the second question, girls changed the rule: multiply the number of triangles by 3. As a general rule they gave: it is needed to multiply the number of triangles by 3 because each triangle has 3 sides. In the case of the second sheet, both girls worked in a similar way. In the table they added 4, and in the second question they multiplied by 4. As a general rule they gave: multiply the number of squares by 4. Two first sheets Nicola filled individually. She did not arrange any figures, at once she moved to the action of filling in proper values in the table. From the very beginning she used correct rules, she was able to generalize them for any element as well. Both in Sylwia and Paulina's solution, and in Nicola's work as well, there was no symbolic notation but only verbal expression of the general rule.

Only during the work with the third sheet, Nicola and Sylwia joined together in one team. Girls cooperated very harmoniously. Nicola allowed Sylwia to solve the task first, and when Sylwia had some trouble with them, Nicola took the initiative and afterwards she told her friend to repeat the whole reasoning. Nicola took the role of a teacher. The teacher was only an observer.

At the beginning Sylwia did not understand what does "in one row" mean. Nicola explained that to her by arranging the pattern with the sticks on the desk and commentating the manner of its development:

1. S: [*reads aloud contents of task*] … what does it mean "in one row"?
2. N: Like this [*she shows by hand a row on the desk*] (…) Look, construct the square [*Sylwia is constructing a square*]. And now you are building the second like this [*Nikola is adding three matches to a square constructed by Sylwia*]. You see, you have three here. And you keep on constructing like that.
3. S: Right
4. N: So in the first one there will be four, and next you will add three
5. S: Aha, I know it now. (…) [*she is starting to fill in the table*] well for one square there will be four, (…) now two times three, because there are three matches here [*she takes away one match from the first square*]
6. N: [*she is adding a match*] now look. There are two squares. We do not add three and three, here there was only four [*she shows the first square*]…
7. S: Seven
8. N: Yes, seven. For this one you add three so that is
9. S: Ten.

Sylwia did not understand the way in which the pattern emerges. Nicola showed the manner of building next squares to her friend. For Nicola only two elements were sufficient to "see" the whole and to understand the general rule of building the pattern. For Sylwia it was too little. After two elements, she did not see the whole structure. The expression "here you have three, and you go on arranging like that" [1] or "next you add three" [3] could have been associated with two previous tasks.

It was adding the same numbers all the time, and it had a reference to the general rule: multiply the number of figures by the number of matches which you have to add. Hence Sylwia applied a solution: for two squares it will be two times three matches – because I am adding three. Only one more explanation by Nicola concerning the manner of arranging and paying attention to the fact that for the first square we use four matches and for each next square only three matches, caused that Sylwia understood the matter of the task and filed the table correctly.

In order to answer the question about 10 squares, the girls "extended" the table, adding by three to the previous number until they received 10 squares. This strategy was very clear and understandable for the girls, and it directly rose from previous established strategy of solving the task. At this stage girls did not discover any dependences between numbers of squares and number of matches which occurred in the pattern.

After having answered the question about 10 squares, Nicola decided to check if the received result is correct. Maybe in this way she wanted to anticipate the teacher's question, as the teacher, after every solution of the task from the previous two sheets stated questions such as: "why will it be that result?", "how do you know that it will be like this?". And maybe she intuitively used, postulated by G. Poly (, "a glance backward", and by this she showed a great mathematical maturity. Verification of correctness of the result happened as follows:

10. N: … Ten times four, or full squares, it will be forty. And now not all were with four matches, nine were with three, weren't they?
11. S: Yes
12. N: So, then we subtract these nine matches, that is thirty one, a kind of a verification.
13. S: Right.

While checking if the result is correct, Nicola referred to the previous task and to the manner of forming the pattern in that task. Therefore, she connected two different, separate experience that led her to create a different thinking model from the executed procedure. Her reasoning looked as following: I have to built ten squares; if these would be separated squares, for each one I need to use four matches, that is 4 x 10 = forty matches.

But my squares have to be connected so only the first will be made of four matches and for each next one I need only three ones, that is one match less than it was. I will build nine of these "incomplete" squares, that is for nine times, I will have one match less. This way of understanding turned out to be very helpful in further work with the task and benefited with discovering some interesting dependences. Here, girls used different connections: they used the separation rule. Focusing attention on the property of the operations, in order to "use arithmetic in a correct way" caused that the girls forgot about the structure of the pattern and occurring dependences.

While answering the question about 25 squares, girls used a following strategy (a well known for us: additive function property): 25 that is 2x10 and 5. For 10 squares we need 31 matches, what we know from the task 2a). So 20 squares are 2x31 that is 62 matches. Moreover, I add 5 squares that is 5 x 3 matches, so in total I have 77 matches. While making a verification, girls notice a mistake:

14. N: There are 25 squares, times 4 [*Sylwia counts on calculator*] that is 100. But yet, we must subtract 24 (…) [*she reads the result from calculator*] 76. Why did we make a mistake?

Girls want to find the source of their mistake. For this purpose, they counted the task again, using the calculator. But they repeated that reasoning which led them to a mistake:

15. N: So, well, it will be like this [*she counts on calculator*]: 10 x 4 = 40, subtract 9 equals 31 [*she checks the result on the sheet*]. We made it right. Now 31 x 2 = 62 (…) And now we still have to add these 5 squares that is 5 x 3 so that is 15.
16. Ex: And why do you add two times 31?
17. N: So that were 20 squares.
18. Ex: Well, these 31 squares will form one row composed of 10 squares, won't it?
19. N: Yes
20. Ex: And for this you add the second row of 10 squares, don't you? To the first row of squares we add the second one in order to create one long row composed of 20 squares. (…) And as you have already arranged the row of 10 squares and beside it the second one of 10 squares, and if you moved them closer now …
21. N: So, we have to take one match away… Aha [*she corrects the mistake*] that is right, 76. Ok. Well that is 161. So let's do it like this: [*she writes on the sheet*] 161 x 4
22. S: [*she counts on the calculator*]
23. N: Ok., and now … [*she takes the calculator and makes calculations 644–160*] …484

Now as a justification Nicola used the following reasoning: from 484 matches I subtract 4 – because this much is needed for the first square. Now I divide 480 by 3 – because the rest of squares have three matches. I get 480: 3–160. These are 160 incomplete squares plus that first one, which gives us 161.

The strategy, which up to this time was only the way of justification of the result, became the rule according to which students were solving the task. It turned out that this rule is reliable – it allowed to find the mistake during the previous task. Besides, it is clear and comprehensible. It is very easy to apply, it is applicable for all examples:

24. Ex: And if you had to arrange a thousand squares?
25. N: It is easy. Look. We write one thousand, don't we? [*she writes 1000*] As we did it here and we multiply by 4. (…) And from these four thousand what do we need to do?
26. S: Subtract
27. N: Subtract this one match from every square except the first one
28. S: Well. That means we have to subtract three
29. N: Why three? (…) Look, you have 1000 squares, you multiply them by each of this kind [*she points out the arranged square with four matches*] that is by four, as if each one had four sides, right? And you have four … no, it can be done differently
30. S: Four thousand.
31. N: One thousand multiplied by three, that will be three thousand.
32. S: Yes
33. N: And now what do we do? You add one square, that one made of four. This way seems easier to me.

During the conversation Nicola noticed another dependence occurring between the number of squares and the number of matches. She succeeded in noticing this thanks to using each time the procedure of "verification" for the correctness of the result. This is a different dependence, which does not reflect the way of the pattern creation. Although, for us this is evident, for a child it is difficult to notice. In order to discover it, the student has to make a "division" of the first square between 1 + 3. S/he must see the square not as a whole, unitary and static figure, but as a dynamic creation. In this case the pattern is built "from the end" that is of incomplete squares (I take three matches, add next three ones, later next three, etc), and after having built the whole, it "closes" the first square, adding the lacking side.

The general rule for that task which was given by the girls goes: multiply the number of squares by three and add one.

Summary:

In the carried out research, students coped with the new task very well. They were able to perceive the dependences occurring in the task, they used the noticing rules correctly. While solving the task, students were using the set of information which they succeeded to gather earlier. They used information about a square and information about the shape of the pattern. These data allowed them to make a general model – "generic model" (Hejný, Kratochvílová 2004) concerning the development of next puzzle elements.. They were able to generalize the dependence discovered by them and to give the dependence's oral record. So they were able to wander off from the concrete facts and start their abstract thinking.

The part of research which was presented here, shows, as far as ways of thinking are concerned, how different children are. One can also see that the verbalization of their own thoughts is very important during the process of solving the task. So is the verification of their own progress of reasoning – the verification of the already obtained results. G. Polya called this "a glance backwards". While checking the correctness of the task execution, students discovered a new approach to the task. Therefore, through words' verification they were able to change their view on the task. The verbalization caused the change in their activity. Thanks to it, they were able to generalize the perceiving rule.

The most important thing for attaining the success while solving tasks of the whole series was the ability to create both: connections among experience assembled on previous levels, and connections with other pieces of mathematical knowledge. The first task gave the chance to make the generalization, to search for the answer to the question about the number of matches needed for building separate figures. That experience was helpful in the second task – in general, students transferred the strategy from the task about squares to the task about triangles without any problems. The girls from the team described above, used the same procedure.

The strategy of counting the amount of elements for the pattern consisting of 10 connected squares concerned the transfer of the way of work from the task about separate squares. But on the level of checking this task, a new discover developed, which was not a verbalization of the applied procedure.

The next important moment which caused new discoveries was the shift from a thinking procedure of counting matches in the pattern in the third task, to applying the rule of divisibility of multiplication with regard to adding. While counting the amount of matches for the pattern built of 25 squares, girls automatically moved to the strategy of counting matches for 20 squares (understood as 2 x 10) and 5 squares. That strategy turn out to be ineffective, but the fact that it was applied by students is worth emphasizing. The necessity of changing the correctness of obtained results forced girls to verify that strategy and to made them search for new connections. This time, the experience from the first and the second task was used constructively.

The girls presented different levels of knowledge. Regardless of that fact, both of them were able to find themselves in an "algebra reality". Both showed that they are able to think algebraically despite of the fact, that they became familiar with arithmetic on different levels. Simultaneously, the arithmetic and extensive focusing on calculations and proper usage of arithmetic rules hindered the correct approach to the solution, shaded or covered the matter of the task. So maybe it is worth developing algebraic thinking regardless of the arithmetic one?

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Preparing of pupils for notion of limits

In this article we present our results concerning qualitative and quantitative research carried out at the St Andrew secondary school in Ruzomberok in October 2007. The research was aimed at the relationships between input and output factors in the teaching process. In the article we describe the results of input test.

MESC: H30, E30

Introduction

The following two notions are included in the senior year curriculum at secondary schools: *limit of a sequence* and *sum of an infinite series*. We realized an experimental teaching devoted to understanding of these notions by students. Our experimental group consisted of students of the St Andrew secondary school in Ruzomberok.

The goal of the research was also to analyze the students' mistakes and to find their roots. The problems we have solved with students are usually not contained in typical mathematical textbooks. Similar research has been described in Hejný, Michalcová, 2001 and Sierpinska, 1987. In this article we analyze input test.

Input test

We assumed that the students had preliminary knowledge from some other areas of school mathematics. Based on our previous qualitative research, we have considered the following input factors: L – logic, AV – algebraic terms, PN – understanding of infinity, N – inequalities. Accordingly, the students wrote the following input test:

Variant A

1. (factor L) Negate the next propositions:

a) There exists a state in which every law is at least two times revised. (3 points)

b) For each natural number x there exists a natural number y such that $x+y=5$. (3 points)

c) All numbers are even. (2 points)

2. (factor AV) Simplify the following expressions (specify conditions):

a) $\dfrac{u^2-4}{u+2}$ (3 points)

b) $\dfrac{u^3-8}{u-2}$ (3 points)

c) $\left(\frac{u^2-4}{u+2}+\frac{u^3-8}{u-2}\right):(u+1)$ (4 points)

d) $\left(\frac{u^2 v^{\frac{1}{3}}}{u^{\frac{3}{2}} v^3}\right)^{\frac{4}{3}}$ (3 points)

3. (factor PN) Answer the following questions:
   a) Is the set of natural numbers greater than one milliard finite or infinite? (1 point)
   b) Which number is the greatest? (1 point)
   c) Let n be a natural number. Is the expression $2n$ increasing or decreasing in n? If it is increasing, find the greatest number, which we can get. If it is decreasing, find the smallest number, which we can get. (2 points)
   d) Let n be a natural number. Is $\frac{1}{n}$ increasing in n or it is decreasing? If it is increasing, find the greatest number, which we can get. If it is decreasing, find the smallest number, which we can get. (2 points)
   e) Given a line in a plane, how many parallel lines do there exist? (1 point)
   f) In the plain, let A be a point and let p be a line which does not contain the point A. How many lines containing the point A and parallel to the line p do exist in the plane? (1 point)
   g) How many points does contain a line segment, which is 10 centimeter long? (1 point)
4. (Factor N) In the set of real numbers, solve the next systems of inequalities:
   a) $4x + 6 < 2x + 5 < 5x + 8$, (5 points)
   b) $2x + 7 < 4x - 5 < x + 6$. (5 points)

Variant B consisted of similar examples. The points obtained by students in corresponding example was the value of the factor. Because we need values from interval $\langle 0,1\rangle$, the values were normalized for statistical processing.

The topic of the factors L, AV is a part of the thematic unit *Introduction to study of mathematics* in the curriculum for first year of secondary schools in Slovakia (see Curriculum, 1997). The topic of the factor N is a part of the thematic unit *Functions, equations and inequalities* also for first year of secondary schools. The factor PN was devoted to the intuitive understanding of infinity.

Frequent mistakes of students

In example 1a) we find next mistakes:

– Wrong negation of first quantifier: *"There does not exist any state, in which every law is at least two times revised." (Zuzana)*. The students negated only the first quantifier, but they haven't the negation of complex proposition.
– Wrong negation of second quantifier: *"In each state is every law at most one times revised." (Júlia)*

In example 1b) some students negated correctly the quantifier, but forgot negate the propositional form $x + y = 5$. Another group use the wrong terms $\forall x \notin$ N or $\exists y \notin$ N. Sometimes the students handled formulas only formally, not understanding the content.

In example 1c) we can find compositions of mistakes from 1a) and 1b). One group of students wrote *“There does not exist any number, which is even” (Zuzana)*. The second group of students wrote: *“There exists a number which is odd” (Mária)* or *“There exists a number which is even” (Andrea)*.

In example 2 a lot of students had problems with term $u^3 - 8$. They tried to simplify this term similarly as $u^2 - 4$:

Petra: $\frac{u^3 - 8}{u - 2} = \frac{(u-2)(u+2)(u+2)}{u-2} = (u+2$

Barbora used the wrong equation $(u + 3)^2 = u^2 + 3u + 9$. She exchange it with equation $(u + 3)^2 = u^2 + 6u + 9$.

Many students did not solve example 2d. Some students had problems with powers and with simplifying of terms. Katarina wrote $u^3 = 3u$, Lucia L. $\left(\frac{u^2 v^{\frac{1}{3}}}{u^{\frac{3}{2}} v^3}\right)^{\frac{4}{3}} = \frac{(u^2)^{\frac{4}{3}} \left(v^{\frac{1}{3}}\right)^{\frac{4}{3}}}{u^{\frac{3}{2}} v^3}$

and Lucia S. $\frac{u^{\frac{4}{9}}}{u^{\frac{3}{4}}} = u^{\frac{4}{9}:\frac{3}{4}}$.

In example 3 the typical mistake was that the line segment has two points and there is no line in a plane, which contains the point A and it is parallel to the line p. Some students wrote that the greatest number is infinity ∞.

In example 4 some students used a right algorithm for solution, but they did not find a correct result. For example, Katarína wrote $x < \frac{3}{2}$ and $x > -\frac{5}{3}$ and the result was $x \in \left(\frac{3}{2}; \infty\right)$ because she drew incorrect picture for this result:

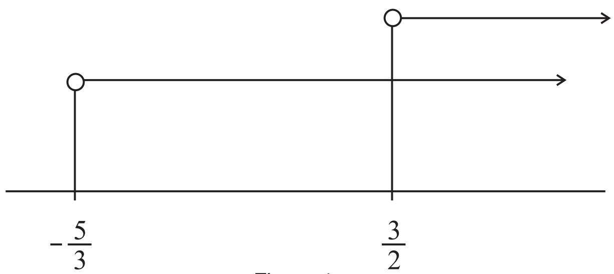

Figure 1

Some students did not multiply correctly the both sides of inequality with a negative number:

Ján: $-3x < 5$ Jana: $-3x < 3$

$x < -\frac{5}{3}$ $x > 1$

Marek had problems with equivalent simplifying: $2x + 5 < 5x + 8$

$$7x > -3$$

We can resume this students' mistakes in the next table:

Factor L	Factor AV	Factor PN	Factor N
Wrong negation of first or second quantifier Forgetting of negation of the propositional form Using wrong terms $\forall x \notin$ N or $\exists y \notin$ N	Using wrong equations of the type $a^3 - b^3$ problems with powers and with simplifying of terms with powers	Infinity as a number line segment has two points	Bad solutions of logical composition of the propositional forms (inequalities) Bad multiplication of inequalities with negative numbers Bad equivalent simplifying

Table 1

Some results of quantitative research

The quantitative research was oriented to the relationships between input factors. We analyze now the next hypotheses:

Ha: The factors AV and L influence to factor N.

Hb: The factors AV and L do not influence to factor N.

The correlation indexes are shown in next table:

	AV	L	N
AV	1	0,53	0,34
L		1	0,41
N			1

Table 2. Correlation indexes

In the experimental group we had 54 students and the critical value of correlation index is $r_{54}(0,05) \approx 0,279$. Factors AV and L correlate with factor N, because the correlation indexes are more than the critical value ($0,34 > 0,279$ and $0,41 > 0,279$).

The influence of the factors can be better seen from the implicative graph prepared by software CHIC:

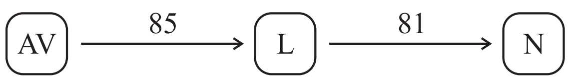

Figure 2. Implicative graph

The values in the figures are implicative indexes, which have values between 0 and 100 percent. These indexes show how strong is the implicative relationship between factors.

Correlative analysis and implicative graph supports hypothesis *Ha*, so we can reject the hypothesis *Hb*. Interestingly, there is strong influence of factor AV to the factor N through factor L.

Conclusions

The results of next qualitative research show that the students have problems with solving examples dealing with the limits of sequences and the sum of infinite series and the problems are conditioned by the lack of knowledge of previous parts of school mathematics (solving of inequalities, simplifying of the algebraic terms – part of mathematics, which is taught in the firs year of secondary school in Slovakia). This show also the pupils' solutions of the input test.

The results of quantitative research show that factors AV and N have influence to the factor N. That means the successful solution of inequalities depends from the ability of pupils simplify the algebraic terms and their logical knowledge. Other researches (see Gunčaga, 2004; Tkačik, 2004; Vancsó, 2003) follows the results of this research. Another problems for propedeutics of calculus teaching in the first year of secondary school is possible to find in Wachnicki, Powązka, 2002 and Zhouf, Sykora, 2002.

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